

# Painlevé 2 equation with arbitrary monodromy parameter, topological recursion and determinantal formulas

K. Iwaki<sup>\*</sup> and O. Marchal<sup>†</sup>

<sup>\*</sup> *Graduate School of Mathematics, Nagoya University, Japan* <sup>1</sup>

<sup>†</sup> *Université de Lyon, CNRS UMR 5208, Université Jean Monnet, Institut Camille Jordan, France* <sup>2</sup>

**Abstract:** The goal of this article is to prove that the determinantal formulas of the Painlevé 2 system identify with the correlation functions computed from the topological recursion on their spectral curve for an arbitrary non-zero monodromy parameter  $\theta$ . The result is established for a WKB expansion of two different Lax pairs associated to the Painlevé 2 system, namely the Jimbo-Miwa Lax pair and the Harnad-Tracy-Widom Lax pair, where a small expansion parameter  $\hbar$  is introduced by a proper rescaling. The proof is based on showing that these systems satisfy the topological type property introduced in [2] and [3]. In the process, we explain why the insertion operator method traditionally used to prove the topological type property is currently incomplete and we propose new algebraic methods to bypass the issue. Eventually, taking the time parameter  $t$  to infinity we observe that the symplectic invariants  $F^{(g)}$  computed from the Hermite-Weber curve  $y^2 = \frac{1}{4}(x^2 - 4\theta)$  and the Bessel curve  $y^2 = \frac{x+\theta^2}{4x^2}$  are equal to respectively  $\mp \frac{B_{2g}}{2g(2g-2)\theta^{2g-2}}$  for  $g \geq 2$ . This result is surprising since both spectral curves are not trivially related (no symplectic transformation between them) and could open the way to new interesting identities. Our work generalizes similar results obtained from random matrix theory in the special case of vanishing monodromies ( $\theta = 0$ ). Explicit computations up to  $g = 3$  are provided along the paper as an illustration of the results.

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<sup>1</sup>iwaki@math.nagoya-u.ac.jp

<sup>2</sup>olivier.marchal@univ-st-etienne.fr

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## 1. INTRODUCTION

In the past decade, the connection between random matrix theory, topological recursion and integrable systems has been developed intensively. Indeed, it was first proved that the partition function describing hermitian random matrix models (first one matrix models and later two matrix models) are isomonodromic tau-functions [5, 6], a central element of integrable systems. Additionally, the local statistics of eigenvalues in hermitian matrix models have been proved to be universal and related to Fredholm determinants whose kernels are determined by the nature of the point in the global distribution (edge, bulk point, critical points, etc.) [21, 22]. Lately, these Fredholm determinants were expressed with some Painlevé transcendents [22]. Recently, Eynard and Orantin provided a recursive algorithm, known as “the topological recursion” [12], to compute the (possibly formal)  $\frac{1}{N}$  expansion of the correlation functions and partition function of any hermitian matrix model. This recursion was generalized almost immediately to any “spectral curve” that may or may not come from a matrix model [12]. This topological recursion has been proved very useful in enumerative geometry where many combinatorial results were recently obtained or rediscovered with this formalism [1, 9, 10, 11]. In particular the main interest of the topological recursion is the definition of a series of numbers  $F^{(g)}$  known as “symplectic invariants” that are invariant under symplectic transformations of the initial spectral curve and that reconstruct the logarithm of the partition function when the spectral curve arises from a matrix model. In a more recent article, Bergère and Eynard [3] were able to associate a natural spectral curve to any  $2 \times 2$  Lax pair and provided some determinantal formulas attached to the Lax pair. These determinantal formulas match the correlation functions and symplectic invariants obtained from the computation of the topological recursion on the spectral curve when some additional conditions, known generically as the “topological type” (TT) property, are satisfied. Eventually, they gave 3 sufficient conditions to prove that a given Lax pair is of topological type. More recently, these notions were extended successfully to  $n \times n$  Lax pairs by Bergère, Borot and Eynard [2]. These results are important since they can be used to prove that the determinantal formulas and the tau-function of the Lax pair can be computed perturbatively to any order with the topological recursion associated to the spectral curve, which in general is relatively easy. So far, the TT conditions have been proved in three different cases:

- First in [4], in relation with the local statistics of eigenvalues near the edge of the distribution for a hermitian matrix model, the authors proved the TT property for the Painlevé 2 system (with the Jimbo-Miwa Lax pair) with vanishing monodromy. The approach was generalized in the case of a critical edge with the  $(2m, 1)$  hierarchy in [19]. These results were recently recovered and precised in [7, 8].
- In [20], in relation with local statistics of eigenvalues in the bulk of the distribution for a hermitian matrix model, the authors proved that the result holds for the Painlevé 5 system with vanishing monodromy parameters.
- Eventually in [2], the authors were able to prove the TT property for the  $q$ -th reduction of the KP hierarchy, that is to say all  $(p, q)$  models. In particular this includes the Painlevé 1 equation (for which there is no monodromy parameter).

However it is worth mentioning that all these articles use at some point the method of the insertion operator. However in this article we show that the current proof regarding the insertion operator is incomplete and we present another way based on loop equations to bypass this issue (See Appendix C).

In this article, our main goal is to prove that the TT property holds for the Painlevé 2 equation:

$$(1.1) \quad \hbar^2 \ddot{q} = 2q^3 + tq - \theta + \frac{\hbar}{2}$$

(where  $\dot{\phantom{x}} = \frac{d}{dt}$  and  $\hbar$  is a small parameter) with arbitrary monodromy parameter  $\theta \neq 0$ . More precisely, we will prove the result for two different ( $\hbar$ -deformed) Lax pairs frequently used to describe the Painlevé 2 system: the Jimbo-Miwa (JM) Lax pair and the Harnad-Tracy-Widom (HTW) Lax pair. (See [14, 16].) These two Lax pairs are related by a Laplace-type integral transformation ([17]). Although these Lax pairs describe the same integrable system (Painlevé 2), their spectral curves are totally different and are not symplectically equivalent. For these two Lax pairs we will review how to insert formally a small expansion parameter  $\hbar$  and how to produce the spectral curve and the tau-function. Then, after presenting the topological recursion and the determinantal formulas, we will prove the TT property by proving the three conditions proposed in [3].

This result proves that the generating functions for both sets of symplectic invariants  $F_{\text{JM}}^{(g)}(t)$  and  $F_{\text{HTW}}^{(g)}(t)$  defined from the spectral curves of JM pair and HTW pair give the corresponding tau-functions of Painlevé 2 (see Theorem 3.2 and 4.2). Note that, although  $F_{\text{JM}}^{(g)}(t)$  and  $F_{\text{HTW}}^{(g)}(t)$  are different since the two spectral curves are not symplectically equivalent, both of them gives tau-functions. As presented in Theorem 3.3 and 4.3 the connection between the two sets of symplectic invariants appears from constant terms (a tau-function is always defined up to a constant) that are fixed by the topological recursion. More specifically one of our main results is that:

$$(1.2) \quad F_{\text{JM}}^{(g)}(t) = F_{\text{HTW}}^{(g)}(t) + \frac{B_{2g}}{2g(2g-2)\theta^{2g-2}} \quad \text{for } g \geq 2.$$

Along the article  $B_{2g}$  stands for the Bernoulli numbers<sup>3</sup> We explain why the specific constant terms appear and we connect them to two simple spectral curves: the Hermite-Weber curve (semi-circle curve) for which the result has been known for a long time and the Bessel curve for which we could not find any reference in the literature.

## 2. SUMMARY OF THE MAIN RESULTS

This article aims at a better understanding between the integrable structure of the Painlevé equations and the topological recursion. Our main results are:

- We prove that the determinantal formulas and tau-functions associated to two different Painlevé 2 Lax pairs (JM and HTW) are identical to the correlation functions and symplectic invariants computed by the topological recursion applied to the corresponding spectral curves (Theorem 5.1).
- Explicit results for the expansion of the tau-function are presented for our two Lax pairs and provides a non-trivial identity when studying their large  $t$  behavior (Theorem 4.3 and (4.29)). This also prove the relation (1.2) between two symplectic invariants for JM and HTW spectral curves.
- New methods of proof of the TT property are introduced in this article (Appendix B and C). In particular the presence of a compatible time differential system and of a genus 0 curve with even zeros are shown to be of critical importance to the TT property. Moreover these new methods can be easily applied to more general situations and should provide a way to perform the same analysis for the other Painlevé equations.

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<sup>3</sup> Bernoulli numbers are defined by:

$$\frac{t}{e^t - 1} - 1 + \frac{t}{2} = \sum_{m=1}^{\infty} B_{2m} \frac{t^{2m}}{(2m)!}.$$

For example,  $B_2 = \frac{1}{6}$ ,  $B_4 = -\frac{1}{30}$ ,  $B_6 = \frac{1}{42}$ .

- We show that the method of the insertion operator used in several papers is incomplete since there is a subtle gap in the proof (Appendix C.1). This issue was the main reason for the introduction of new methods to prove the TT property.

### 3. JIMBO-MIWA LAX PAIR FOR PAINLEVÉ 2

In this section we show how to introduce a formal parameter  $\hbar$  into the Jimbo-Miwa Lax pair and we present the differential systems as well as the tau-function and its expansion in  $\hbar$ . Finally we compute the spectral curve and we illustrate our results with the computation of the first leading terms of the tau-function and the symplectic invariants  $F^{(g)}$  of the curve.

**3.1. Introduction of the  $\hbar$  parameter in the Jimbo-Miwa Lax pair.** The following  $2 \times 2$  Lax pair is equivalent (up to a certain gauge transformation) to the one given in Appendix C of [16]:

$$(3.1) \quad \begin{cases} \frac{\partial \Psi}{\partial x}(x, t) &= \begin{pmatrix} x^2 + p + \frac{t}{2} & x - q \\ -2(xp + qp + \theta) & -(x^2 + p + \frac{t}{2}) \end{pmatrix} \Psi(x, t), \\ \frac{\partial \Psi}{\partial t}(x, t) &= \begin{pmatrix} \frac{x+q}{2} & \frac{1}{2} \\ -p & -\frac{x+q}{2} \end{pmatrix} \Psi(x, t). \end{cases}$$

The Lax pair we will use in this article is a  $\hbar$ -deformed version of (3.1). The introduction of  $\hbar$  is done by the following rescaling of all quantities involved in the former Lax pair.

$$(3.2) \quad x \rightarrow \hbar^{-\frac{1}{3}} \tilde{x}, \quad p \rightarrow \hbar^{-\frac{2}{3}} \tilde{p}, \quad t \rightarrow \hbar^{-\frac{2}{3}} \tilde{t}, \quad q \rightarrow \hbar^{-\frac{1}{3}} \tilde{q}, \quad \theta \rightarrow \hbar^{-1} \tilde{\theta}$$

and provides the  $\hbar$ -deformed Lax pair (we omit  $\sim$  for clarity):

$$(3.3) \quad \begin{cases} \hbar \frac{\partial \Psi}{\partial x}(x, t) &= \begin{pmatrix} x^2 + p + \frac{t}{2} & x - q \\ -2(xp + qp + \theta) & -(x^2 + p + \frac{t}{2}) \end{pmatrix} \Psi(x, t) \stackrel{\text{def}}{=} \mathcal{D}(x, t) \Psi(x, t), \\ \hbar \frac{\partial \Psi}{\partial t}(x, t) &= \begin{pmatrix} \frac{x+q}{2} & \frac{1}{2} \\ -p & -\frac{x+q}{2} \end{pmatrix} \Psi(x, t) \stackrel{\text{def}}{=} \mathcal{R}(x, t) \Psi(x, t). \end{cases}$$

We call the Lax pair (3.3) the Jimbo-Miwa pair (JM pair, for short).

We remind the reader that  $q$  and  $p$  are implicitly assumed to depend on the time variable  $t$  (and also on  $\hbar$ ) but not on  $x$ . Moreover  $\theta$  is independent of  $x, t$  and  $\hbar$ . The parameter  $\theta$  is called the monodromy parameter. Throughout of the paper, we assume that

$$(3.4) \quad \theta \neq 0.$$

The compatibility equations of the differential system (also known as zero-curvature equations) are given by:

$$(3.5) \quad \hbar \left( \frac{\partial \mathcal{D}}{\partial t} - \frac{\partial \mathcal{R}}{\partial x} \right) + [\mathcal{D}, \mathcal{R}] = 0.$$

From (3.3) they are equivalent to:

$$(3.6) \quad \hbar \dot{p} = -2qp - \theta, \quad \hbar \dot{q} = p + q^2 + \frac{t}{2}.$$

Here and in what follows a dot is used to denote the derivative relatively to  $t$  when no ambiguity appears. Differentiating the last equation and eliminating the  $p(t)$  function with the first equation gives that  $q(t)$  satisfies the Painlevé 2 equation (with the small parameter  $\hbar$ ):

$$(3.7) \quad \hbar^2 \ddot{q} = 2q^3 + tq - \theta + \frac{\hbar}{2}.$$

This type of Painlevé equations with a small parameter  $\hbar$  was studied in [18].

In this paper we are interested in the  $\hbar$ -perturbative expansion of a solution of Painlevé 2:

$$(3.8) \quad q(t) = \sum_{k=0}^{\infty} q_k(t) \hbar^k = q_0(t) + q_1(t) \hbar + q_2(t) \hbar^2 + \dots.$$

The top term  $q_0(t)$  satisfies

$$(3.9) \quad 2q_0(t)^3 + tq_0(t) - \theta = 0 \text{ and } \dot{q}_0(t) = -\frac{q_0(t)^2}{4q_0(t)^3 + \theta}.$$

As is clear from (3.9),  $\dot{q}_0(t)$  has singularity where  $4q_0(t)^3 + \theta = 0$  holds. Such a point on  $t$ -plane is called a turning point of Painlevé 2 in [18]. In what follows, we assume that  $t$  lies on a domain on which  $4q_0(t)^3 + \theta \neq 0$  holds. Note also that  $q_0(t) \neq 0$  for any  $t$  under the assumption (3.4).

Since  $q_0(t)$  is a solution of a cubic equation, there are 3 possible choices of the branches for  $q_0(t)$ . In particular when  $t \rightarrow \infty$  there are three possible behaviors depending on the chosen branch:

$$(3.10) \quad q_0(t) \underset{t \rightarrow \infty_A}{\sim} \frac{\theta}{t}, \quad q_0(t) \underset{t \rightarrow \infty_B}{\sim} \sqrt{\frac{-t}{2}}, \quad q_0(t) \underset{t \rightarrow \infty_C}{\sim} -\sqrt{\frac{-t}{2}}.$$

It is easy to see that, once we fix the branch of the algebraic function  $q_0(t)$ , the coefficients  $q_i(t)$ 's appearing in (3.8) are determined recursively. Thanks to the relation (3.6), we also get a similar  $\hbar$ -expansion of  $p(t)$ .

**3.2. Hamiltonian system and tau-function of the JM Lax pair.** The tau-function are classically defined since the works of Jimbo-Miwa-Ueno [15] from which an alternative formulation was also presented in [7, §1.5.3]. For the JM pair, these quantities are easy to derive (the leading order of the matrices  $\mathcal{D}(x, t)$  and  $\mathcal{R}(x, t)$  when  $x \rightarrow \infty$  are both diagonal) and can be directly adapted from the known  $\hbar = 1$  case. The Hamiltonian system attached to the JM pair (3.3) is:

$$(3.11) \quad \begin{cases} \hbar \dot{q} &= \frac{\partial H_{\text{JM}}}{\partial p} = p + q^2 + \frac{t}{2}, \\ \hbar \dot{p} &= -\frac{\partial H_{\text{JM}}}{\partial q} = -2qp - \theta, \end{cases}$$

where  $H_{\text{JM}}$  is the Hamiltonian for Painlevé 2:

$$(3.12) \quad H_{\text{JM}} = \frac{1}{2}p^2 + \left(q^2 + \frac{t}{2}\right)p + \theta q.$$

Let  $\sigma(t)$  be the corresponding Hamiltonian function, that is, the function obtained by substituting a solution  $(q, p)$  of (3.11) into  $H_{\text{JM}}$ . It satisfies:

$$(3.13) \quad \dot{\sigma}(t) = \frac{p}{2} \text{ and } \hbar \ddot{\sigma}(t) = -qp - \frac{\theta}{2}.$$

as well as the  $\sigma$ -form of the Painlevé 2 equation:

$$(3.14) \quad (\hbar \ddot{\sigma})^2 + 4(\dot{\sigma})^3 + 2t(\dot{\sigma})^2 - 2\sigma\dot{\sigma} - \frac{\theta^2}{4} = 0.$$

Then, the tau-function for JM Lax pair is defined by:

$$(3.15) \quad -\hbar^2 \frac{d}{dt} \ln \tau_{\text{JM}} = \sigma(t).$$

Since (3.14) only involves even power of  $\hbar$  then the  $\hbar$ -expansion of  $\sigma(t)$  and  $\ln \tau(t)$  only involve even powers of  $\hbar$ :

$$(3.16) \quad \sigma(t) = \sum_{k=0}^{\infty} \sigma_{2k}(t) \hbar^{2k},$$

$$(3.17) \quad \ln \tau_{\text{JM}} \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \tau_{2k}(t) \hbar^{2k-2}, \quad \tau_{2k}(t) = -\int^t \sigma_{2k}(s) ds.$$

Moreover, (3.13) implies that  $p(t)$  also only contains even order terms:

$$(3.18) \quad p(t) = \sum_{k=0}^{\infty} p_{2k}(t) \hbar^{2k}.$$

**3.3. First orders of the JM tau-function.** In this section we present the computation of the first orders of the tau-function for the JM Lax pair. In what follows, we choose to express all quantities as function of  $q_0(t)$  which is a solution of (3.9). Straightforward computations give:

$$\begin{aligned}
(3.19) \quad \sigma_0(t) &= \frac{\theta(8q_0^3 - \theta)}{8q_0^2}, \\
\sigma_2(t) &= -\frac{\theta q_0}{8(4q_0^3 + \theta)^2}, \\
\sigma_4(t) &= -\frac{3\theta q_0^4(560q_0^6 - 184\theta q_0^3 + 3\theta^2)}{32(4q_0^3 + \theta)^7}, \\
\sigma_6(t) &= -\frac{\theta q_0^7(3203200q_0^{12} - 3668064\theta q_0^9 + 838632\theta^2 q_0^6 - 39482\theta^3 q_0^3 + 189\theta^4)}{32(4q_0^3 + \theta)^{12}}.
\end{aligned}$$

In particular one can also verify directly that the coefficients presented here satisfy the differential equation (3.14). Integrating over  $t$  leads to:

$$\begin{aligned}
(3.20) \quad \tau_0(t) &= \frac{4}{3}\theta q_0^3 + \frac{\theta^3}{24q_0^3} + \frac{\theta^2}{2}\ln q_0 + \text{Cste}, \\
\tau_2(t) &= \frac{1}{24}\ln(1 + \frac{\theta}{4q_0^3}) + \frac{1}{12}\ln 2 + \text{Cste}, \\
\tau_4(t) &= \frac{\theta(700q_0^6 - 85\theta q_0^3 - 2\theta^2)}{480(4q_0^3 + \theta)^5} + \text{Cste}, \\
\tau_6(t) &= \frac{\theta(6726720q_0^{15} - 5017712\theta q_0^{12} + 541132\theta^2 q_0^9 - 1089\theta^3 q_0^6 + 160\theta^4 q_0^3 + 4\theta^5)}{4032(4q_0^3 + \theta)^{10}} + \text{Cste}.
\end{aligned}$$

Here the constant terms are to be understood as not depending on  $t$ . We will see that these integration constants are specified by the topological recursion, and correspond to lower end-points for the integral (3.17) defining  $\tau_{2k}$  taken at  $t = \infty$  for  $k \geq 2$ . Actually, we can choose  $\infty$  (for any  $\infty_A, \infty_B$  and  $\infty_C$ ) as the lower end-point since  $\sigma_{2k} \underset{t \rightarrow \infty}{=} O(t^{-2})$  holds if  $k \geq 2$ . This property can easily be proved by a recursion relation satisfied by  $q_i(t)$ 's appearing in (3.8).

**3.4. Spectral curve and topological recursion for the JM pair.** From Bergère and Eynard [3] we know that the spectral curve of a Lax pair is given by the leading order in  $\hbar$  of the characteristic polynomial of  $\mathcal{D}(x, t)$ . Thus we find:

$$(3.21) \quad y^2 = (x - q_0)^2(x^2 + 2q_0x + 3q_0^2 + t) = (x - q_0)^2 \left( x + q_0 - \sqrt{-\frac{\theta}{q_0}} \right) \left( x + q_0 + \sqrt{-\frac{\theta}{q_0}} \right)$$

where  $q_0(t)$  is the solution of (3.9). We call (3.21) the Jimbo-Miwa spectral curve (JM curve, for short). JM curve is of genus 0 with two branchpoints. It can be parametrized with a global Zhukovsky variable:

$$(3.22) \quad \begin{cases} x(z) = -q_0 + \frac{1}{2}\sqrt{-\frac{\theta}{q_0}} \left( z + \frac{1}{z} \right), \\ y(z) = \frac{1}{2}\sqrt{-\frac{\theta}{q_0}} \left( z - \frac{1}{z} \right) \left( -2q_0 + \frac{1}{2}\sqrt{-\frac{\theta}{q_0}} \left( z + \frac{1}{z} \right) \right). \end{cases}$$

With this parametrization, the branchpoints are located at  $z = \pm 1$  and the differential  $ydx$  has two poles at  $z = 0$  and  $z = \infty$ .

**Definition 3.1** (Definition 4.2 of [12]). For  $g \geq 0$  and  $n \geq 1$ , define the Eynard-Orantin differential (or correlation function)  $\omega_n^{(g)}(z_1, \dots, z_n)$  of the type  $(g, n)$  for the spectral curve (3.22) by the

following topological recursion relation ([12]):

$$\begin{aligned}
(3.23) \quad \omega_1^{(0)}(z_1) &= y(z_1)dx(z_1), \\
\omega_2^{(0)}(z_1, z_2) &= \frac{dz_1 dz_2}{(z_1 - z_2)^2}, \\
\omega_{n+1}^{(g)}(z_0, z_1, \dots, z_n) &= \sum_{r \in R} \text{Res}_{z=r} K(z_0, z) \left[ \omega_{n+1}^{(g-1)}(z, \bar{z}, z_1, \dots, z_n) \right. \\
&\quad \left. + \sum_{\substack{g_1+g_2=g \\ I \cup J = \{1, \dots, n\}}} \omega_{1+|I|}^{(g_1)}(z, z_I) \omega_{1+|J|}^{(g_2)}(\bar{z}, z_J) \right].
\end{aligned}$$

Here  $R$  is the set of branchpoints,  $\bar{z}$  is the local conjugate of  $z$  near a branchpoint  $r$ ,

$$(3.24) \quad K(z_0, z) = \frac{\int_{\bar{z}}^z \omega_2^{(0)}(\cdot, z_0)}{(y(z) - y(\bar{z}))dx(z)}$$

is called the recursion kernel, and the summation in the last line means “except for the cases  $(g_1, I) = (0, \emptyset)$  and  $(g_2, J) = (0, \emptyset)$ ”.

For JM curve (3.22),  $R = \{+1, -1\}$  and  $\bar{z} = z^{-1}$  for both branchpoints  $r = \pm 1$ . The Eynard-Orantin differential  $\omega_n^{(g)}$ 's are meromorphic multi-differentials on the  $n$ -times product of spectral curve, and known to be holomorphic except for the branch points if  $(g, n) \neq (0, 1), (0, 2)$ . Since our spectral curve has genus 0, the topological recursion becomes easier (see [12] for general case).

We also introduce symplectic invariants  $F^{(g)}$  for the spectral curve, following [12]:

**Definition 3.2** (Definition 4.3 of [12]). The  $g$ -th symplectic invariant of the spectral curve is defined by

$$(3.25) \quad F^{(g)} = \frac{1}{2-2g} \sum_{r \in R} \text{Res}_{z=r} \Phi(z) \omega_1^{(g)}(z) \quad \text{for } g \geq 2$$

where

$$(3.26) \quad \Phi(z) = \int_{z_o}^z y(z)dx(z) \quad (z_o \text{ is a generic point}).$$

$F^{(0)}$  and  $F^{(1)}$  are defined in alternative manner (see §4.2.2 and §4.2.3 of [12]).

Denote by  $F_{\text{JM}}^{(g)}$  the symplectic invariants for JM curve (3.21). Then, we find:

$$\begin{aligned}
(3.27) \quad F_{\text{JM}}^{(0)} &= \frac{4\theta}{3}q_0^3 + \frac{\theta^3}{24q_0^3} + \frac{\theta^2}{2}\ln(q_0) - \frac{\theta^2}{4} - \frac{\theta^2}{2}\ln(\theta) + \theta^2\ln(2), \\
F_{\text{JM}}^{(1)} &= -\frac{1}{24}\ln\left(\theta^2\left(1 + \frac{\theta}{4q_0^3}\right)\right), \\
F_{\text{JM}}^{(2)} &= \frac{1}{480} \frac{(2048q_0^{12} + 2560\theta q_0^9 + 1280\theta^2 q_0^6 + 1020\theta^3 q_0^3 - 45\theta^4)q_0^3}{\theta^2(4q_0^3 + \theta)^5}, \\
F_{\text{JM}}^{(3)} &= -\frac{q_0^6}{4032\theta^4(\theta + 4q_0^3)^{10}} \left( 4194304q_0^{24} + 10485760\theta q_0^{21} + 11796480\theta^2 q_0^{18} \right. \\
&\quad \left. + 7864320\theta^3 q_0^{15} + 3440640\theta^4 q_0^{12} - 5694528\theta^5 q_0^9 + 5232752\theta^6 q_0^6 - 510412\theta^7 q_0^3 + 3969\theta^8 \right).
\end{aligned}$$

Moreover, it is easy to prove that when  $q_0 \rightarrow 0$  (i.e.  $t \rightarrow \infty_A$ ) the correlation functions  $\omega_n^{(g)}$  and the symplectic invariants  $F_{\text{JM}}^{(g)}$  (identified with  $\omega_0^{(g)}$  in the next formula) behave like:

$$(3.28) \quad \omega_n^{(g)}(z_1, \dots, z_n) \underset{q_0 \rightarrow 0}{\sim} \text{Cste } q_0^{\frac{3}{2}(2g-2+n)} dz_1 \cdots dz_n.$$



Indeed, the kernel  $K(z_0, z)$  used in the recursion behaves like:

$$(3.29) \quad K(z_0, z) = -\frac{4z^4}{(z^2 - 1)(z_0 z - 1)(z_0 - z) \left( (1 + z) \left( \frac{-\theta}{q_0} \right)^{\frac{3}{2}} + 4\theta z \right)} \frac{dz_0}{dz} = O\left(q_0^{\frac{3}{2}}\right).$$

Thus adding a power  $q_0^{\frac{3}{2}}$  at each step of the recursion. In particular, we get:

$$(3.30) \quad \lim_{t \rightarrow \infty_A} F_{\text{JM}}^{(g)}(t) = 0 \quad \text{for } g \geq 2.$$

**3.5. Tau-function and symplectic invariants for the JM pair.** In this subsection we state one of our main results regarding the relationship between symplectic invariants  $F_{\text{JM}}^{(g)}$  and the tau-function of Painlevé 2.

**Theorem 3.1.** *The JM pair is of topological type (in the sense of Section 5) and we have:*

$$(3.31) \quad \frac{dF_{\text{JM}}^{(g)}(t)}{dt} = -\sigma_{2g}(t) \quad \text{for } g \geq 2.$$

Theorem 3.1 will be proved in Section 5 and in Appendix. From former results (3.19), (3.20) and (3.27) we can verify that  $\frac{dF_{\text{JM}}^{(g)}(t)}{dt} = -\sigma_{2g}(t)$  also hold for  $g = 0$  and  $g = 1$ . This implies the following:

**Theorem 3.2.** *The generating function of symplectic invariants of JM curve (3.21) gives a tau-function of Painlevé 2. In other words,*

$$(3.32) \quad \ln \tau_{\text{JM}} = \sum_{g=0}^{\infty} \hbar^{2g-2} F_{\text{JM}}^{(g)}(t)$$

satisfies (3.15). Furthermore, we have

$$(3.33) \quad F_{\text{JM}}^{(g)}(t) = -\int_{\infty_A}^t \sigma_{2g}(s) ds \quad \text{for } g \geq 2.$$

Theorem 3.2 follows from Theorem 3.1 and (3.30).

**3.6. Limit at  $\infty_B$ : The Hermite-Weber curve.** We already know that the functions  $F_{\text{JM}}^{(g)}(t)$  vanish for  $g \geq 2$  when  $t \rightarrow \infty_A$  (i.e.  $q_0 \rightarrow 0$ ). We find that some interesting numbers appear when taking the limit of  $F_{\text{JM}}^{(g)}(t)$  to  $t \rightarrow \infty_{B,C}$ . Unfortunately taking  $q_0 \rightarrow \infty$  in the spectral curve (3.21) is not directly possible since it leads to a singular curve (a perfect square) for which the topological recursion is not well defined. To avoid this difficulty, we perform a symplectic transformation of the curve for which we know that the  $F^{(g)}$  are invariant for  $g \geq 2$ . Let us perform the following transformation:

$$(3.34) \quad x = \frac{1}{2\sqrt{-q_0}} X - q_0, \quad y = 2\sqrt{-q_0} Y$$

giving the curve:

$$(3.35) \quad Y^2 = \left( \frac{X^2}{4} - \theta \right) \left( 1 + \frac{X}{4(-q_0)^{\frac{3}{2}}} \right)^2.$$

In the limit  $t \rightarrow \infty_{B,C}$  (i.e.,  $q_0 \rightarrow \infty$ ), we obtain the Hermite-Weber curve (also known as semi-circle curve):

$$(3.36) \quad Y^2 = \frac{X^2}{4} - \theta$$



which can be parametrized into:

$$(3.37) \quad \begin{cases} X(z) = \sqrt{\theta} \left( z + \frac{1}{z} \right), \\ Y(z) = \frac{\sqrt{\theta}}{2} \left( z - \frac{1}{z} \right). \end{cases}$$

This curve corresponds to the Gaussian Hermitian Matrix model and up to a trivial normalization corresponds to the so-called semi-circle law. The  $g$ -th symplectic invariant for this curve coincide with the Euler characteristics of the moduli space of genus  $g$  Riemann surfaces, which is written in terms of the Bernoulli numbers ([13]). In fact in [21] it is proved that the symplectic invariants  $F_{\text{Weber}}^{(g)}$  for the spectral curve (3.37) are given by

$$(3.38) \quad \begin{aligned} F_{\text{Weber}}^{(0)} &= \frac{3\theta^2}{4} - \frac{\theta^2}{2} \ln \theta, \\ F_{\text{Weber}}^{(1)} &= -\frac{1}{12} \ln \theta, \\ F_{\text{Weber}}^{(g)} &= -\frac{B_{2g}}{2g(2g-2)\theta^{2g-2}} \quad \text{for } g \geq 2. \end{aligned}$$

Since it is known from [12] that the symplectic invariants (and correlation functions) obtained for a limiting curve are equal to the limit of symplectic invariants, we have the following:

**Theorem 3.3.** *The limit  $t \rightarrow \infty_{B,C}$  of the symplectic invariant  $F_{\text{JM}}^{(g)}(t)$  of JM curve is given by*

$$(3.39) \quad \lim_{t \rightarrow \infty_{B,C}} F_{\text{JM}}^{(g)}(t) = -\frac{B_{2g}}{2g(2g-2)\theta^{2g-2}} \quad \text{for } g \geq 2.$$

In particular taking  $q_0 \rightarrow 0$  in (3.27) we can verify that this result holds for  $g = 2$  and  $g = 3$ . Theorem 3.3 and equation (3.33) also imply the following:

$$(3.40) \quad \int_{\infty_A}^{\infty_{B,C}} \sigma_{2g}(t) dt = \frac{B_{2g}}{2g(2g-2)\theta^{2g-2}} \quad \text{for } g \geq 2.$$

Note that a path connecting  $\infty_A$  and  $\infty_{B,C}$  never exists when  $\theta = 0$  since the equation (3.9) defining  $q_0(t)$  splits in that case. This is consistent with the fact that the r.h.s. of the last equation blows up when  $\theta \rightarrow 0$ .

#### 4. THE HARNAD-TRACY-WIDOM LAX PAIR

In this section, we develop the same approach for the Painlevé 2 system but with another Lax pair as a starting point. This Lax pair is not trivially connected to the previous one and is also commonly used to describe some properties of the Painlevé 2 system. To our knowledge the two Lax pairs (up to trivial transformations) studied in this article represent the two usual pairs used to describe the Painlevé 2 system.

**4.1. Introduction of the  $\hbar$  parameter in the Harnad-Tracy-Widom Lax pair.** The following Lax pair is introduced in [14]:

$$(4.1) \quad \begin{cases} \frac{\partial}{\partial x} \Psi(x, t) &= \begin{pmatrix} -q(t) + \frac{\theta}{2x} & x - p(t) - 2q^2(t) - t \\ \frac{1}{2} + \frac{p(t)}{2x} & q(t) - \frac{\theta}{2x} \end{pmatrix} \Psi(x, t), \\ \frac{\partial}{\partial t} \Psi(x, t) &= \begin{pmatrix} q(t) & -x \\ -\frac{1}{2} & -q(t) \end{pmatrix} \Psi(x, t). \end{cases}$$

A relationship between (3.1) and (4.1) is investigated in [17]. Like in the JM case we can introduce a small expansion parameter  $\hbar$  with a suitable rescaling of the variables:

$$(4.2) \quad x \rightarrow \hbar^{-\frac{2}{3}} \tilde{x}, \quad q \rightarrow \hbar^{-\frac{1}{3}} \tilde{q}, \quad p \rightarrow \hbar^{-\frac{2}{3}} \tilde{p}, \quad t \rightarrow \hbar^{-\frac{2}{3}} \tilde{t}, \quad \theta \rightarrow \hbar^{-1} \tilde{\theta}$$

as well as a suitable gauge transformation  $\Psi \rightarrow \text{diag}(\hbar^{-\frac{1}{6}}, \hbar^{\frac{1}{6}})\Psi$ . Omitting  $\sim$  for clarity, we get:

$$(4.3) \quad \begin{cases} \hbar \frac{\partial}{\partial x} \Psi(x, t) &= \begin{pmatrix} -q(t) + \frac{\theta}{2x} & x - p(t) - 2q^2(t) - t \\ \frac{1}{2} + \frac{p(t)}{2x} & q(t) - \frac{\theta}{2x} \end{pmatrix} \Psi(x, t) \stackrel{\text{def}}{=} \mathcal{D}(x, t) \Psi(x, t), \\ \hbar \frac{\partial}{\partial t} \Psi(x, t) &= \begin{pmatrix} q(t) & -x \\ -\frac{1}{2} & -q(t) \end{pmatrix} \Psi(x, t) \stackrel{\text{def}}{=} \mathcal{R}(x, t) \Psi(x, t). \end{cases}$$

We call the Lax pair (4.3) Harnad-Tracy-Widom pair (HTW pair, for short).

The compatibility equations for this Lax pair are given by:

$$(4.4) \quad \hbar \dot{q} = q^2 + p + \frac{t}{2} \quad \text{and} \quad \hbar \dot{p} = -2qp - \theta.$$

As in the Jimbo-Miwa case, we recover that  $q(t)$  is a solution of the Painlevé 2 equation:

$$(4.5) \quad \hbar^2 \ddot{q} = 2q^3 + tq - \theta + \frac{\hbar}{2}.$$

Although the compatibility equations are the same as for the JM pair, the definition of Jimbo-Miwa-Ueno tau-function is a little different ([15, 16]; see also appendix D). The Hamiltonian for the HTW pair is:

$$(4.6) \quad H_{\text{HTW}} = \frac{1}{2}p^2 + \left(q^2 + \frac{t}{2}\right)p + \theta q + \frac{t^2}{8}$$

and the tau-function for HTW pair is defined as

$$(4.7) \quad -\hbar^2 \frac{d}{dt} \ln \tau_{\text{HTW}} = H_{\text{HTW}} = \sigma(t) + \frac{t^2}{8}$$

where  $\sigma(t)$  is the solution of the  $\sigma$ -form of Painlevé 2 in (3.14).

**4.2. Spectral curve and topological recursion for the HTW pair.** As usual the spectral curve is given by the leading order in  $\hbar$  of the characteristic polynomial of  $\mathcal{D}(x, t)$ . We find:

$$(4.8) \quad y^2 = \frac{1}{2x^2} \left(x - \frac{\theta}{2q_0}\right)^2 (x + 2q_0^2)$$

where  $q_0(t)$  is a solution of (3.9). It is a genus 0 curve with a single branchpoint arising at  $x = -2q_0^2$  but with a pole singularity at  $x = 0$ . It can be parametrized globally with:

$$(4.9) \quad \begin{cases} x(z) = 2q_0^2(z^2 - 1), \\ y(z) = \frac{z(\theta - 4q_0^3(z^2 - 1))}{4q_0^2(z^2 - 1)} = \frac{z(q_0 x(z) - \frac{\theta}{2})}{x(z)}. \end{cases}$$

We call this spectral curve the HTW spectral curve. Note that in the Eynard-Orantin language, the local conjugate point around the branchpoint (here it is a global involution) is given by  $\bar{z} = -z$ . With this parametrization, the branchpoint is located at  $z = 0$  and  $y(z)$  has two pole singularities at  $z = \pm 1$ . We define the Eynard-Orantin differentials and the symplectic invariants  $F_{\text{HTW}}^{(g)}$  for the HTW spectral curve (4.9) in the same way as (3.23) and (3.25). Note that there is no symplectic transformations between the HTW curve and the JM curve since they provide different  $F^{(g)}$  (the signs and constant terms are different). We find:

$$(4.10) \quad \begin{aligned} F_{\text{HTW}}^{(0)}(t) &= \frac{q_0^6}{3} + \frac{5\theta}{6}q_0^3 + \frac{\theta^2}{2} \ln(q_0) + \frac{3}{4}\theta^2 \ln(2) - \frac{3\theta^2}{4} + \frac{i\pi\theta^2}{4}, \\ F_{\text{HTW}}^{(1)}(t) &= \frac{1}{24} \ln\left(1 + \frac{\theta}{4q_0^3}\right) - \frac{1}{48} \ln(2), \\ F_{\text{HTW}}^{(2)}(t) &= \frac{\theta(700q_0^6 - 85\theta q_0^3 - 2\theta^2)}{480(4q_0^3 + \theta)^5}, \end{aligned}$$

$$F_{\text{HTW}}^{(3)}(t) = \frac{\theta (6726720q_0^{15} - 5017712\theta q_0^{12} + 541132\theta^2 q_0^9 - 1089\theta^3 q_0^6 + 160\theta^4 q_0^3 + 4\theta^5)}{4032 (4q_0^3 + \theta)^{10}}.$$

Additionally, as  $t \rightarrow \infty_{B,C}$  (i.e.  $q_0 \rightarrow \infty$ ), it is easy to prove by recursion that the correlation functions and symplectic invariants (identified with  $\omega_0^{(g)}$  in the next formula) generated by the topological recursion on (4.9) behave like:

$$(4.11) \quad \omega_n^{(g)}(z_1, \dots, z_n) \underset{q_0 \rightarrow \infty}{\sim} \text{Cste } q_0^{-3(2g-2+n)} dz_1 \cdots dz_n.$$

Indeed, the recursion kernel behaves like

$$(4.12) \quad K(z_0, z) = \frac{(z^2 - 1)}{2z(z^2 - z_0^2)(\theta - 4q_0^3(z^2 - 1))} \frac{dz_0}{dz} = O(q_0^{-3}).$$

Thus adding  $q_0^{-3}$  at each step of the recursion. In particular for  $n = 0$  we find that:

$$(4.13) \quad \lim_{t \rightarrow \infty_{B,C}} F_{\text{HTW}}^{(g)}(t) = 0 \quad \text{for } g \geq 2.$$

**4.3. Tau-function and symplectic invariants for the HTW pair.** In this section we state the second main result.

**Theorem 4.1.** *The HTW pair is of topological type (in the sense of Section 5) and we have:*

$$(4.14) \quad \frac{dF_{\text{HTW}}^{(g)}(t)}{dt} = -\sigma_{2g}(t) \quad \text{for } g \geq 2.$$

In particular we can verify that (4.14) is correct for  $g = 2$  and  $g = 3$ . Proof of Theorem 4.1 will be given in Section 5 and appendix. We can also verify that

$$(4.15) \quad \frac{dF_{\text{HTW}}^{(0)}(t)}{dt} = -\left(\sigma_0(t) + \frac{t^2}{8}\right), \quad \frac{dF_{\text{HTW}}^{(1)}(t)}{dt} = -\sigma_2(t)$$

holds in accordance with (4.7). This leads to the following theorem:

**Theorem 4.2.** *The generating function of symplectic invariants of the HTW curve (4.8) gives a  $\tau$ -function of Painlevé 2. In other words,*

$$(4.16) \quad \ln \tau_{\text{HTW}} = \sum_{g=0}^{\infty} \hbar^{2g-2} F_{\text{HTW}}^{(g)}(t)$$

satisfies (4.7). Furthermore, we have:

$$(4.17) \quad F_{\text{HTW}}^{(g)}(t) = -\int_{\infty_{B,C}}^t \sigma_{2g}(s) ds \quad \text{for } g \geq 2.$$

**4.4. Limit at  $t = \infty_A$ : the Bessel curve.** Let us realize the following symplectic transformation on the spectral curve (4.8):

$$(4.18) \quad x = \frac{2q_0^2}{\theta^2} X, \quad y = \frac{\theta^2}{2q_0^2} Y$$

we get the new spectral curve:

$$(4.19) \quad Y^2 = \frac{(X + \theta^2) \left(1 - \frac{4q_0^3}{\theta^3} X\right)^2}{4X^2}.$$

When  $t \rightarrow \infty_A$  (i.e.  $q_0 \rightarrow 0$ ) we get that the limiting curve becomes the Bessel curve:

$$(4.20) \quad Y^2 = \frac{X + \theta^2}{4X^2}$$

which can be parametrized into:

$$(4.21) \quad \begin{cases} X(z) = \theta^2(z^2 - 1), \\ Y(z) = \frac{z}{2\theta(z^2 - 1)}. \end{cases}$$

In particular straightforward computations of the topological recursion gives:

$$(4.22) \quad \begin{aligned} F_{\text{Bessel}}^{(0)} &= -\frac{3\theta^2}{4} + \frac{\theta^2}{2} \ln 2 + \frac{\theta^2}{2} \ln \theta + \frac{i\pi}{24}, \\ F_{\text{Bessel}}^{(1)} &= \frac{i\pi\theta^2}{24} + \frac{1}{24} \ln 2 + \frac{1}{12} \ln \theta, \\ F_{\text{Bessel}}^{(2)} &= -\frac{1}{240\theta^2}, \\ F_{\text{Bessel}}^{(3)} &= \frac{1}{1008\theta^4}. \end{aligned}$$

General properties regarding limits and symplectic transformations of the curve in the topological recursion tell us that:

$$(4.23) \quad \lim_{t \rightarrow \infty_A} F_{\text{HTW}}^{(g)}(t) = F_{\text{Bessel}}^{(g)} \quad \text{for } g \geq 2.$$

On the other hand, it follows from (3.40) and (4.17) that we have

$$(4.24) \quad \lim_{t \rightarrow \infty_A} F_{\text{HTW}}^{(g)} = - \int_{\infty_{B,C}}^{\infty_A} \sigma_{2g}(t) = \frac{B_{2g}}{2g(2g-2)\theta^{2g-2}} \quad \text{for } g \geq 2.$$

Therefore, as a corollary of our main theorems, we have computed  $F_{\text{Bessel}}^{(g)}$  explicitly:

**Theorem 4.3.** *The symplectic invariants  $F_{\text{Bessel}}^{(g)}$  of the Bessel curve (4.21) are given by*

$$(4.25) \quad F_{\text{Bessel}}^{(g)} = \frac{B_{2g}}{2g(2g-2)\theta^{2g-2}} \quad \text{for } g \geq 2.$$

To our knowledge, (4.25) has not been mentioned in the literature.

Remark: We note that the Hermite-Weber curve and the Bessel curve provide the same symplectic invariants  $F^{(g)}$  for  $g \geq 2$  up to a global minus sign. The last remark can be made more specific. Let us parametrize the Bessel curve in following way:

$$(4.26) \quad x_{\text{Bessel}}(s) = -\frac{\theta^2}{4} \left( s + \frac{1}{s} \right)^2, \quad y_{\text{Bessel}}(s) = -\frac{i \left( s - \frac{1}{s} \right)}{\theta \left( s + \frac{1}{s} \right)^2}$$

and use the following symplectic transformation:

$$(4.27) \quad x_1 = \frac{i\sqrt{\theta}}{2} \ln x_{\text{Bessel}} + \frac{i\sqrt{\theta}}{2} \ln \left( \frac{-\theta^2}{4} \right), \quad y_1 = -\frac{2i}{\sqrt{\theta}} x_{\text{Bessel}} y_{\text{Bessel}}$$

to get a new spectral curve sharing the same symplectic invariants as the one produced from the Bessel curve:

$$(4.28) \quad x_1(s) = i\sqrt{\theta} \ln \left( s + \frac{1}{s} \right), \quad y_1(s) = \frac{\sqrt{\theta}}{2} \left( s - \frac{1}{s} \right).$$

It is now tempting to compare it with the previous parametrization (3.37) of the Hermite-Weber spectral. The similarity of the two curves is striking but the presence of a logarithm in  $x_1(s)$  makes them impossible to be connected by any symplectic transformation. This would require to define  $x_1(s) = i\sqrt{\theta} \ln \frac{X(s)}{\sqrt{\theta}}$  while keeping  $y_1(s) = Y(s)$ . It is unclear for us to see if this little change is

always accountable for just a change of sign in the symplectic invariants  $F^{(g)}$  or if this observation may have some geometric interpretation. In terms of tau-functions it means that we have:

$$(4.29) \quad \tau_{\text{Bessel}} \tau_{\text{Weber}} = 1$$

up to a proper normalization of the tau-functions. The general form of the relation makes us think that it might be related to some kind of supersymmetric models where sometimes partition functions happen to be constant. From the topological recursion perspective, we do not know any general transformations on the spectral curve that would only change the signs of all  $F^{(g)}$  and we let it here as an open problem.

## 5. DETERMINANTAL FORMULAS AND TOPOLOGICAL TYPE PROPERTY

In this section we review the determinantal formulas formalism and the issue of the topological type property. Then we mention our main results and discuss about the consequences. The proof of the topological type properties are postponed in Appendix A, B and C.

Here we remind the reader about determinantal formulas developed in [3]. We indicate also that these results have been generalized for higher dimensional Lax pairs in [2] where the connection with isomonodromic tau-functions was also clarified.

Determinantal formulas are built from the differential equation with respect to  $x$  in the JM or HTW Lax pair:

$$(5.1) \quad \hbar \frac{d}{dx} \Psi(x) = \mathcal{D}(x) \Psi(x), \quad \Psi(x) = \begin{pmatrix} \psi(x) & \phi(x) \\ \tilde{\psi}(x) & \tilde{\phi}(x) \end{pmatrix}$$

where the matrix  $\mathcal{D}(x)$  is traceless and has a formal series expansion

$$(5.2) \quad \mathcal{D}(x) = \sum_{k=0}^{\infty} D^{(k)}(x) \hbar^k.$$

There exists a matrix-type WKB formal solution

$$(5.3) \quad \Psi(x) = \left( \sum_{k=0}^{\infty} \Psi^{(k)}(x) \hbar^k \right) \exp \left( \frac{T(x)}{\hbar} \right), \quad T(x) = \text{diag}(s(x), -s(x))$$

which is normalized by  $\det \Psi = 1$ . Here the phase function  $s(x)$  satisfies

$$(5.4) \quad \frac{d}{dx} s(x) = \sqrt{E_{\infty}(x)},$$

where

$$(5.5) \quad E_{\infty}(x) = -\det \mathcal{D}^{(0)}(x) = \begin{cases} (x - q_0)^2 (x^2 + 2q_0 x + 3q_0^2 + t) & \text{for JM case,} \\ \frac{1}{2x^2} \left( x - \frac{\theta}{2q_0} \right)^2 (x + 2q_0^2) & \text{for HTW case.} \end{cases}$$

Determinantal formulas are obtained from the Christoffel-Darboux kernel:

$$(5.6) \quad K(x_1, x_2) = \frac{\psi(x_1) \tilde{\phi}(x_2) - \tilde{\psi}(x_1) \phi(x_2)}{x_1 - x_2}$$

with the following definition:

**Definition 5.1.** The (connected) correlation functions are defined by:

$$(5.7) \quad \begin{aligned} W_1(x) &= \frac{d\psi}{dx}(x) \tilde{\phi}(x) - \frac{d\tilde{\psi}}{dx}(x) \phi(x), \\ W_n(x_1, \dots, x_n) &= -\frac{\delta_{n,2}}{(x_1 - x_2)^2} + (-1)^{n+1} \sum_{\sigma: n\text{-cycles}} \prod_{i=1}^n K(x_i, x_{\sigma(i)}) \quad \text{for } n \geq 2. \end{aligned}$$

The correlation functions are formal power series of  $\hbar$  whose coefficients are symmetric functions of  $x_1, \dots, x_n$ . Note that there exists an alternative expression for the correlation functions [3]. Define the rank 1 projector by

$$(5.8) \quad M(x, t) = \Psi(x, t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Psi^{-1}(x, t) = \begin{pmatrix} \psi\tilde{\phi} & -\psi\phi \\ \tilde{\psi}\tilde{\phi} & -\phi\tilde{\psi} \end{pmatrix}.$$

It is in fact the canonical projector on the first coordinate taken into the basis defined by  $\Psi(x, t)$ . The rank 1 projector satisfies:

$$M^2 = M, \quad \text{Tr } M = 1, \quad \det M = 0.$$

Theorem 2.1 of [3] gives an alternative expression for  $W_n(x_1, \dots, x_n)$ :

$$(5.9) \quad \begin{aligned} W_1(x) &= -\frac{1}{\hbar} \text{Tr} (\mathcal{D}(x, t) M(x)), \\ W_2(x_1, x_2) &= \frac{\text{Tr} (M(x_1) M(x_2)) - 1}{(x_1 - x_2)^2}, \\ W_n(x_1, \dots, x_n) &= (-1)^{n+1} \text{Tr} \sum_{\sigma: n\text{-cycles}} \prod_{i=1}^n \frac{M(x_{\sigma(i)})}{x_{\sigma(i)} - x_{\sigma(i+1)}} \\ &= \frac{(-1)^{n+1}}{n} \sum_{\sigma \in S_n} \frac{\text{Tr } M(x_{\sigma(1)}) \dots M(x_{\sigma(n)})}{(x_{\sigma(1)} - x_{\sigma(2)}) \dots (x_{\sigma(n-1)} - x_{\sigma(n)})} \quad \text{for } n \geq 3. \end{aligned}$$

Now we give the definition of the topological type property for the differential equation (5.1) having the spectral curve of genus 0.

**Definition 5.2** (Definition 3.3 of [2], Section 2.5 of [3]). The differential equation (5.1) is said to be of topological type if the correlation functions  $W_n(x_1, \dots, x_n)$  defined in Definition 5.1 satisfy the following three conditions:

- (1) Parity property:  $W_n|_{\hbar \mapsto -\hbar} = (-1)^n W_n$  hold for  $n \geq 1$ .
- (2) Existence of a topological expansion: The leading order of the series expansion of the correlation function  $W_n$  is at least of order  $\hbar^{n-2}$ :  $W_n = O(\hbar^{n-2})$ . When these two conditions are satisfied,  $W_n$  has the following expansion (called topological expansion):

$$(5.10) \quad W_n(x_1, \dots, x_n) = \sum_{g=0}^{\infty} \hbar^{2g-2+n} W_n^{(g)}(x_1, \dots, x_n) \quad \text{for } n \geq 1.$$

- (3) Pole structure: The correlation functions  $W_n(x_1, \dots, x_n)$  have no poles at even zeros of  $E_{\infty}(x)$  given in (5.5).

**Remark**: If the spectral curve  $y^2 = E_{\infty}(x)$  has a genus  $h \geq 1$ , then we must impose an additional condition on filling fractions (period integral of  $W_n^{(g)}$  along closed cycles in the spectral curve). See Section 2.5 of [3].

The authors of [2] and [3] proved that:

**Proposition 5.1** (Theorem 2.1 of [3], Theorem 3.1 and Corollary 4.2 of [2]). *If the differential equation (5.1) is of topological type, then the coefficients  $W_n^{(g)}(x_1, \dots, x_n)$  appearing in the expansion (5.10) of the correlation function  $W_n(x_1, \dots, x_n)$  are identical to the correlation functions  $\omega_n^{(g)}$  obtained from the topological recursion applied on the spectral curve  $y^2 = E_{\infty}(x)$  in the following way:*

$$(5.11) \quad W_n^{(g)}(x(z_1), \dots, x(z_n)) dx(z_1) \dots dx(z_n) = \omega_n^{(g)}(z_1, \dots, z_n) \quad \text{for } g \geq 0 \text{ and } n \geq 1,$$

where  $x(z)$  is the rational function of  $z$  appearing in the parametrization (3.22) or (4.9) of the spectral curve. Moreover, if  $\mathcal{D}(x)$  depends on an isomonodromic time  $t$ , then the  $\hbar$  expansion of the

isomonodromic tau-function in the sense of [15] of the system matches with the generating function of symplectic invariants  $F^{(g)}$  obtained from the topological recursion applied to  $y^2 = E_\infty(x)$ .

Our context fits perfectly with the previous proposition and thus only the proof of the topological type property remains. Our main theorem (including the statement of Theorem 3.2 and 4.2) claims that the conditions (1)  $\sim$  (3) hold for both Lax pairs:

**Theorem 5.1.** *For any choice of the monodromy parameter  $\theta \neq 0$ , both of the Jimbo-Miwa Lax pair (3.3) and the Harnad-Tracy-Widom Lax pair (4.3) are of topological type. Therefore, the  $\hbar$ -expansion of the tau-function  $\tau_{2g}$  and correlation functions  $W_n^{(g)}$  respectively identify with the symplectic invariants  $F^{(g)}$  and the correlation functions  $\omega_n^{(g)}$  computed from the topological recursion applied to the corresponding spectral curves (3.21) and (4.8) as (5.11).*

We will prove that the three conditions (1)  $\sim$  (3) hold for both Lax pairs in Appendix A  $\sim$  C. Typically when dealing with isomonodromic parameters, the conditions (2) and (3) can be proved using the differential equation with respect to  $t$  (see [20] for example). We will also show how the topological type property identifies the tau-functions and symplectic invariants in appendix D.

## 6. OUTLOOKS

We believe that the topological type property (TT property) should hold for all six Painlevé equations with arbitrary monodromy parameters and any Lax pair as a starting point. In particular in this paper we have introduced new general methods to prove the TT property (See pole structure in appendix B and leading order of the expansion in appendix C) that should generalize easily for other Lax pairs. Our work shows that the absence of monodromy (i.e. taking all monodromy parameters to 0) is not a necessary condition to obtain the TT property. Several natural questions arise from this work:

- Similar computations should be carried for the other Painlevé equations and their standard Lax pairs. We believe that the TT property should hold in all cases but this remains to be checked properly.
- At the conceptual level a better understanding would definitely be an interesting development. Indeed it is even unclear so far to what extent the TT property is connected with Lax pairs. Does it work for any Lax pairs? Only specific integrable systems? Our new methods seem general enough to work for many Lax pairs but a better understanding of the scope and limits of the methods is required.
- On a totally different perspective, we have shown here that studying Lax pairs and their symplectic invariants may lead to explicit formulas for symplectic invariants of new spectral curves. Cases where general formulas for the symplectic invariants are known explicitly are extremely rare and this could provide a way to improve the classification of symplectic invariants for simple spectral curves. This knowledge might be of some use for enumerative geometry where a list of spectral curves and associated symplectic invariants would be helpful.
- In this paper we also draw attention on the incompleteness of the insertion operator method. This calls for a better definition of the insertion operator to fix the current problem.

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## APPENDIX A. PROOF THE PARITY SYMMETRY

We want to prove that the  $\hbar$  series expansion of the determinantal formulas  $W_n(x_1, \dots, x_n)$  only involves powers of  $\hbar$  of the same parity. (Cf., (5.10)). In order to do this, we use Proposition 3.3 of [2] that gives a sufficient criteria to obtain the  $\hbar \leftrightarrow -\hbar$  symmetry. We recall their proposition here:

**Proposition A.1** (Proposition 3.3 of [2]). *Let us denote  $\dagger$  the operator that change  $\hbar$  into  $-\hbar$ . If there exists an invertible matrix  $\Gamma(t)$  independent of  $x$  such that:*

$$(A.1) \quad \Gamma(t) \mathcal{D}^t(x, t) \Gamma^{-1}(t) = \mathcal{D}^\dagger(x, t),$$

*then the correlators  $W_n$  satisfy*

$$(A.2) \quad W_n^\dagger = (-1)^n W_n \quad \text{for } n \geq 1.$$

In particular if this proposition is satisfied then it automatically follows that the  $\hbar$  expansion a given function  $W_n(x_1, \dots, x_n)$  may only involve powers of  $\hbar$  with the same parity (given by the parity of  $n$ ). Therefore all we have to do is prove the existence of a suitable matrix  $\Gamma(t)$  for our two Lax pairs.

Recall that  $\sigma^\dagger = \sigma$  and  $p^\dagger = p$  hold (see (3.16) and (3.18)). Then, it follows from (3.6) that  $q$  satisfies

$$(A.3) \quad q^\dagger = -q - \frac{\theta}{p}.$$

Using this relation we can obtain  $\mathcal{D}^\dagger(x, t)$  and we can find an invertible matrix  $\Gamma$  satisfying (A.1) as follows:

**Theorem A.1.** *We can find suitable matrices  $\Gamma(t)$  for our Lax pairs:*

- *For the Jimbo-Miwa case, the matrix  $\Gamma(t) = \begin{pmatrix} 1 & 0 \\ 0 & -2p(t) \end{pmatrix}$  satisfies (A.1).*
- *For the Harnad-Tracy-Widom case, the matrix  $\Gamma(t) = \begin{pmatrix} 1 & \frac{p(t)}{2\theta} \\ \frac{p(t)}{2\theta} & 0 \end{pmatrix}$  satisfies (A.1).*

*Consequently, the series expansion in  $\hbar$  for  $W_n$  only involves even (resp. odd) powers of  $\hbar$  when  $n$  is even (resp. odd).*

Theorem A.1 follows from (A.3) immediately. In Jimbo-Miwa case we have

$$(A.4) \quad \mathcal{D}^\dagger(x, t) = \begin{pmatrix} x^2 + p + \frac{t}{2} & x + q + \frac{\theta}{p} \\ -2p(x - q) & -(x^2 + p + \frac{t}{2}) \end{pmatrix} \quad \text{and} \quad \mathcal{D}^t(x, t) = \begin{pmatrix} x^2 + p + \frac{t}{2} & -2p \left( x + q + \frac{\theta}{p} \right) \\ x - q & -(x^2 + p + \frac{t}{2}) \end{pmatrix},$$

while in the Harnad-Tracy-Widom case we have:

$$(A.5) \quad \mathcal{D}^\dagger(x, t) = \begin{pmatrix} q + \frac{\theta}{p} + \frac{\theta}{2x} & x - p - 2 \left( q + \frac{\theta}{p} \right)^2 - t \\ \frac{1}{2} + \frac{p}{2x} & - \left( q + \frac{\theta}{p} + \frac{\theta}{2x} \right) \end{pmatrix} \quad \text{and} \quad \mathcal{D}^t(x, t) = \begin{pmatrix} -q + \frac{\theta}{2x} & \frac{1}{2} + \frac{p}{2x} \\ x - p - 2q^2 - t & q - \frac{\theta}{2x} \end{pmatrix}.$$

Then (A.1) can be checked easily by matrix multiplication in both cases. More generally, the parity property can be seen as a byproduct of the fact that the  $\sigma$ -form of Painlevé 2 only involves even powers of  $\hbar$ .

## APPENDIX B. PROOF OF THE POLE STRUCTURE

As mentioned earlier we are interested in this article about solutions  $q(t)$  of Painlevé 2 admitting a formal expansion in the  $\hbar$  parameter. Consequently for both Lax pairs, this implies a series

expansion in  $\hbar$  for  $M(x, t)$ ,  $W_n$ ,  $\mathcal{D}(x, t)$  and  $\mathcal{R}(x, t)$  of the form:

$$\begin{aligned}
(B.1) \quad \mathcal{D}(x, t) &= \sum_{k=0}^{\infty} \mathcal{D}^{(k)}(x, t) \hbar^k \\
\mathcal{R}(x, t) &= \sum_{k=0}^{\infty} \mathcal{R}^{(k)}(x, t) \hbar^k, \\
M(x, t) &= \sum_{k=0}^{\infty} M^{(k)}(x, t) \hbar^k, \\
W_1(x) &= \sum_{k=-1}^{\infty} W_1^{(k)}(x) \hbar^k, \\
W_n(x_1, \dots, x_n) &= \sum_{k=0}^{\infty} W_n^{(k)}(x_1, \dots, x_n) \hbar^k \text{ for } n \geq 2.
\end{aligned}$$

In this appendix an exponent  $^{(k)}$  denotes the coefficient of  $\hbar^k$  in the  $\hbar$ -expansion. Moreover, it is also obvious from the definitions of both Lax pairs that  $\mathcal{D}^{(k)}(x, t)$  and  $\mathcal{R}^{(k)}(x, t)$  do not depend on  $x$  for  $k \geq 1$ .

**B.1. Jimbo-Miwa Lax pair.** We want to prove that the matrices  $M^{(k)}(x, t)$  only have singularities (as a function of  $x$ ) at the branchpoints of the JM spectral curve and possibly a pole at infinity. The plan is first to compute explicitly  $M^{(0)}(x, t)$  and then find a recursive relation between the matrices.

**B.1.1. Computation of  $M^{(0)}(x, t)$ .** Inserting the series expansion of  $M(x, t)$  into the differential system for  $M(x, t)$ :

$$(B.2) \quad \hbar \partial_x M(x, t) = [\mathcal{D}(x, t), M(x, t)] \text{ and } \hbar \partial_t M(x, t) = [\mathcal{R}(x, t), M(x, t)]$$

gives that:

$$(B.3) \quad 0 = [\mathcal{D}^{(0)}(x, t), M^{(0)}(x, t)] \text{ and } 0 = [\mathcal{R}^{(0)}(x, t), M^{(0)}(x, t)].$$

Additionally, since  $M(x, t)$  is a rank 1 projector we know that  $\text{Tr } M(x, t) = 1$  and  $\det M(x, t) = 0$ . At order  $\hbar^0$  this is equivalent to  $\text{Tr } M^{(0)}(x, t) = 1$  and  $\det M^{(0)}(x, t) = 0$ . Since  $M^{(0)}(x, t)_{2,2} = 1 - M^{(0)}(x, t)_{1,1}$ , the second equation of (B.3) only gives 3 different equations:

$$\begin{aligned}
(B.4) \quad 0 &= \frac{1}{2} \left( M^{(0)}(x, t) \right)_{2,1} - \frac{\theta}{2q_0} \left( M^{(0)}(x, t) \right)_{1,2}, \\
0 &= \frac{1}{2} \left( 1 - 2 \left( M^{(0)}(x, t) \right)_{1,1} \right) + (x + q_0) \left( M^{(0)}(x, t) \right)_{1,2}, \\
0 &= (x + q_0) \left( M^{(0)}(x, t) \right)_{2,1} + \frac{\theta}{2q_0} \left( 1 - 2 \left( M^{(0)}(x, t) \right)_{1,1} \right).
\end{aligned}$$

We have used here the fact that  $p_0 = -\frac{\theta}{2q_0}$  and  $t = -2q_0^2 + \frac{\theta}{q_0}$ . It is easy to observe that only two of the previous equations are independent. Therefore we have so far a system of 2 independent equations with 3 unknowns. In order to complete it, we must use the fact that  $\det M^{(0)}(x, t) = 0$ . In the end we find the following system:

$$\begin{aligned}
(B.5) \quad 0 &= \frac{1}{2} \left( M^{(0)}(x, t) \right)_{2,1} - \frac{\theta}{2q_0} \left( M^{(0)}(x, t) \right)_{1,2}, \\
0 &= \frac{1}{2} \left( 1 - 2 \left( M^{(0)}(x, t) \right)_{1,1} \right) + (x + q_0) \left( M^{(0)}(x, t) \right)_{1,2},
\end{aligned}$$

$$0 = \left( M^{(0)}(x, t) \right)_{1,1} \left( 1 - \left( M^{(0)}(x, t) \right)_{1,1} \right) - \left( M^{(0)}(x, t) \right)_{1,2} \left( M^{(0)}(x, t) \right)_{2,1}.$$

It is also important to note that the first equation of (B.3) would have lead exactly to the same system of equations. This system can be solved explicitly and we find:

$$(B.6) \quad M^{(0)}(x, t) = \begin{pmatrix} \frac{1}{2} + \frac{x+q_0}{2\sqrt{(x+q_0)^2 + \frac{\theta}{q_0}}} & \frac{1}{2\sqrt{(x+q_0)^2 + \frac{\theta}{q_0}}} \\ \frac{\theta}{2q_0\sqrt{(x+q_0)^2 + \frac{\theta}{q_0}}} & \frac{1}{2} - \frac{x+q_0}{2\sqrt{(x+q_0)^2 + \frac{\theta}{q_0}}} \end{pmatrix}.$$

It is obvious that  $M^{(0)}(x, t)$  only have singularities at the branchpoints of the spectral curve.

**B.1.2. Recursive system for higher orders.** Let us now look at order  $\hbar^k$  with  $k \geq 1$  in  $[\mathcal{R}^{(0)}(x, t), M^{(0)}(x, t)] = 0$  and in  $\det M(x, t) = 0$ . We get:

$$\begin{aligned} [\mathcal{R}^{(0)}(x, t), M^{(k)}(x, t)] &= \partial_t M^{(k-1)}(x, t) - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)] \\ M^{(k-1)}(x, t)_{1,1} \left( 1 - 2M^{(k-1)}(x, t)_{1,1} \right) &- M^{(0)}(x, t)_{2,1} M^{(k)}(x, t)_{1,2} - M^{(0)}(x, t)_{1,2} M^{(k)}(x, t)_{2,1} \\ &= \sum_{i=1}^{k-1} \left( M^{(i)}(x, t)_{1,1} M^{(k-i)}(x, t)_{1,1} + M^{(i)}(x, t)_{1,2} M^{(k-i)}(x, t)_{2,1} \right). \end{aligned}$$

The first matrix equation provides two independent scalar equations and thus we get a  $3 \times 3$  linear system that can be written in the following matrix form:

$$(B.7) \quad \begin{pmatrix} 0 & -\frac{\theta}{2q_0} & \frac{1}{2} \\ -1 & x+q_0 & 0 \\ x+q_0 & \frac{\theta}{2q_0} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} M^{(k)}(x, t)_{1,1} \\ M^{(k)}(x, t)_{1,2} \\ M^{(k)}(x, t)_{2,1} \end{pmatrix} = \begin{pmatrix} \partial_t M^{(k-1)}(x, t)_{1,1} - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,1} \\ \partial_t M^{(k-1)}(x, t)_{1,2} - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,2} \\ -\sqrt{(x+q_0)^2 + \frac{\theta}{q_0}} \sum_{i=1}^{k-1} \left( M^{(i)}(x, t)_{1,1} M^{(k-i)}(x, t)_{1,1} + M^{(i)}(x, t)_{1,2} M^{(k-i)}(x, t)_{2,1} \right) \end{pmatrix}.$$

Note in particular that the  $3 \times 3$  matrix on the l.h.s. does not depend on the order  $k$  we consider (it is only  $\hbar^0$  terms). In general inverting a matrix may create poles at the zeros of the determinant of the matrix (this is obvious if one uses the definition of the inverse using the matrix of cofactors). However in our case we have:

$$(B.8) \quad \det \begin{pmatrix} 0 & -\frac{\theta}{2q_0} & \frac{1}{2} \\ -1 & x+q_0 & 0 \\ x+q_0 & \frac{\theta}{2q_0} & \frac{1}{2} \end{pmatrix} = -\frac{1}{2} \left( (x+q_0)^2 + \frac{\theta}{q_0} \right).$$

Consequently we get:

$$(B.9) \quad \begin{pmatrix} M^{(k)}(x, t)_{1,1} \\ M^{(k)}(x, t)_{1,2} \\ M^{(k)}(x, t)_{2,1} \end{pmatrix} = \begin{pmatrix} -(x+q_0) & -\frac{\theta}{q_0} & x+q_0 \\ -1 & x+q_0 & 1 \\ 2(x+q_0)^2 + \frac{\theta}{q_0} & \frac{(x+q_0)\theta}{q_0} & \frac{\theta}{q_0} \end{pmatrix} \begin{pmatrix} \frac{1}{(x+q_0)^2 + \frac{\theta}{q_0}} \left( \partial_t M^{(k-1)}(x, t)_{1,1} - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,1} \right) \\ \frac{1}{(x+q_0)^2 + \frac{\theta}{q_0}} \left( \partial_t M^{(k-1)}(x, t)_{1,2} - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,2} \right) \\ -\frac{1}{\sqrt{(x+q_0)^2 + \frac{\theta}{q_0}}} \sum_{i=1}^{k-1} \left( M^{(i)}(x, t)_{1,1} M^{(k-i)}(x, t)_{1,1} + M^{(i)}(x, t)_{1,2} M^{(k-i)}(x, t)_{2,1} \right) \end{pmatrix}.$$

Then since for  $i \geq 1$  we know that  $\mathcal{R}^{(i)}(x, t)$  does not depend on  $x$ , a straightforward induction using (B.6) and (B.9) shows that the only singularities of  $M^{(k)}(x, t)$  are at the branchpoints (solution of  $(x+q_0)^2 + \frac{\theta}{q_0} = 0$ ) and possibly a pole at infinity:

**Theorem B.1.** *For  $k \geq 0$ , the matrices  $M^{(k)}(x, t)$  only have singularities at the branchpoints of the spectral curve  $x = -q_0 \pm \sqrt{\frac{\theta}{q_0}}$  and a possible pole singularity at infinity. In particular they are holomorphic at the even zero of the spectral curve  $x = q_0$ . Consequently the same singularity structure holds for the functions  $W_n^{(g)}(x_1, \dots, x_n)$ .*

Remark: It is also interesting to observe that using the differential equation in  $x$  rather than the one in  $t$  provides a similar set of equations:

$$(B.10) \quad \begin{pmatrix} M^{(k)}(x, t)_{1,1} \\ M^{(k)}(x, t)_{1,2} \\ M^{(k)}(x, t)_{2,1} \end{pmatrix} = \begin{pmatrix} -(x+q_0) & -\frac{\theta}{q_0} & x+q_0 \\ -1 & x+q_0 & 1 \\ 2(x+q_0)^2 + \frac{\theta}{q_0} & \frac{(x+q_0)\theta}{q_0} & \frac{\theta}{q_0} \end{pmatrix} \begin{pmatrix} \frac{1}{2((x+q_0)^2 + \frac{\theta}{q_0})(x-q_0)} \left( \partial_x M^{(k-1)}(x, t)_{1,1} - \sum_{i=0}^{k-1} [\mathcal{D}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,1} \right) \\ \frac{1}{2((x+q_0)^2 + \frac{\theta}{q_0})(x-q_0)} \left( \partial_x M^{(k-1)}(x, t)_{1,2} - \sum_{i=0}^{k-1} [\mathcal{D}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,2} \right) \\ -\frac{1}{\sqrt{(x+q_0)^2 + \frac{\theta}{q_0}}} \sum_{i=1}^{k-1} (M^{(i)}(x, t)_{1,1} M^{(k-i)}(x, t)_{1,1} + M^{(i)}(x, t)_{1,2} M^{(k-i)}(x, t)_{2,1}) \end{pmatrix}.$$

However in this case, it is not possible to exclude pole singularities at the double zero of the spectral curve  $x = q_0$ . Thus we understand here the importance of the differential equation with respect to  $t$  in the context of the determinantal formulas.

**B.2. Harnad-Tracy-Widom Lax pair.** Most of the arguments of the previous section also apply to the Harnad-Tracy-Widom Lax pair.

**B.2.1. Computation of  $M^{(0)}(x, t)$ .** Looking at  $[\mathcal{R}^{(0)}(x, t), M^{(0)}(x, t)] = 0$  and  $\det M^{(0)}(x, t) = 0$  leads to the following system:

$$(B.11) \quad \begin{aligned} 0 &= -xM^{(0)}(x, t)_{2,1} + \frac{1}{2}M^{(0)}(x, t)_{1,2}, \\ 0 &= -x \left( 1 - 2M^{(0)}(x, t)_{1,1} \right) + 2q_0M^{(0)}(x, t)_{1,2}, \\ 0 &= -2q_0M^{(0)}(x, t)_{2,1} + \frac{1}{2} \left( 1 - 2M^{(0)}(x, t)_{1,1} \right), \\ 0 &= M^{(0)}(x, t)_{1,1} \left( 1 - M^{(0)}(x, t)_{1,1} \right) - M^{(0)}(x, t)_{1,2}M^{(0)}(x, t)_{2,1}. \end{aligned}$$

Note that only two of the first three equations are independent. This system of equations admits a unique solution given by:

$$(B.12) \quad M^{(0)}(x, t) = \begin{pmatrix} \frac{1}{2} - \frac{q_0}{\sqrt{2}\sqrt{x+2q_0^2}} & \frac{x}{\sqrt{2}\sqrt{x+2q_0^2}} \\ \frac{1}{2\sqrt{2}\sqrt{x+2q_0^2}} & \frac{1}{2} + \frac{q_0}{\sqrt{2}\sqrt{x+2q_0^2}} \end{pmatrix}.$$

It is then obvious that  $M^{(0)}(x, t)$  only have singularities at  $x = -2q_0^2$  the unique branchpoint of the spectral curve and at infinity. Note also that  $[\mathcal{D}^{(0)}(x, t), M^{(0)}(x, t)] = 0$  would have provided an equivalent system of equations.

**B.2.2. Recursive system for higher orders.** Let us now look at order  $\hbar^k$  with  $k \geq 1$  in  $[\mathcal{R}^{(0)}(x, t), M^{(0)}(x, t)] = 0$  and in  $\det M(x, t) = 0$ . We get:

$$\begin{aligned} [\mathcal{R}^{(0)}(x, t), M^{(k)}(x, t)] &= \partial_t M^{(k-1)}(x, t) - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)] \\ M^{(k-1)}(x, t)_{1,1} \left( 1 - 2M^{(k-1)}(x, t)_{1,1} \right) &- M^{(0)}(x, t)_{2,1}M^{(k)}(x, t)_{1,2} - M^{(0)}(x, t)_{1,2}M^{(k)}(x, t)_{2,1} \\ &= \sum_{i=1}^{k-1} \left( M^{(i)}(x, t)_{1,1}M^{(k-i)}(x, t)_{1,1} + M^{(i)}(x, t)_{1,2}M^{(k-i)}(x, t)_{2,1} \right). \end{aligned}$$

The first matrix equation provides two independent scalar equations and thus we get a  $3 \times 3$  linear system that can be written in the following matrix form:

$$(B.13) \quad \begin{pmatrix} 0 & \frac{1}{2} & -x \\ 2x & 2q_0 & 0 \\ 2q_0 & -\frac{1}{2} & -x \end{pmatrix} \begin{pmatrix} M^{(k)}(x, t)_{1,1} \\ M^{(k)}(x, t)_{1,2} \\ M^{(k)}(x, t)_{2,1} \end{pmatrix} = \begin{pmatrix} \partial_t M^{(k-1)}(x, t)_{1,1} - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,1} \\ \partial_t M^{(k-1)}(x, t)_{1,2} - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,2} \\ \sqrt{2}\sqrt{x+2q_0^2} \sum_{i=1}^{k-1} (M^{(i)}(x, t)_{1,1}M^{(k-i)}(x, t)_{1,1} + M^{(i)}(x, t)_{1,2}M^{(k-i)}(x, t)_{2,1}) \end{pmatrix}.$$

A straightforward computation shows that:

$$(B.14) \quad \det \begin{pmatrix} 0 & \frac{1}{2} & -x \\ 2x & 2q_0 & 0 \\ 2q_0 & -\frac{1}{2} & -x \end{pmatrix} = 2x(x + 2q_0^2).$$

Hence we get:

$$(B.15) \quad \begin{pmatrix} M^{(k)}(x, t)_{1,1} \\ M^{(k)}(x, t)_{1,2} \\ M^{(k)}(x, t)_{2,1} \end{pmatrix} = \begin{pmatrix} -q_0 & \frac{1}{2} & q_0 \\ x & q_0 & -x \\ -\frac{x+4q_0^2}{2x} & \frac{q_0}{2x} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{x+2q_0^2} \left( \partial_t M^{(k-1)}(x, t)_{1,1} - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,1} \right) \\ \frac{1}{x+2q_0^2} \left( \partial_t M^{(k-1)}(x, t)_{1,2} - \sum_{i=0}^{k-1} [\mathcal{R}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,2} \right) \\ \frac{\sqrt{2}}{\sqrt{x+2q_0^2}} \sum_{i=1}^{k-1} (M^{(i)}(x, t)_{1,1} M^{(k-i)}(x, t)_{1,1} + M^{(i)}(x, t)_{1,2} M^{(k-i)}(x, t)_{2,1}) \end{pmatrix}.$$

Hence we conclude the following:

**Theorem B.2.** *For  $k \geq 0$ , the matrices  $M^{(k)}(x, t)$  only have singularities at the branchpoint of the spectral curve  $x = -2q_0^2$  and poles at  $x = 0$  and  $x = \infty$ . In particular they are holomorphic at the even zero of the spectral curve  $x = \frac{\theta}{2q_0}$ . Consequently the same singularity structure holds for the functions  $W_n^{(g)}(x_1, \dots, x_n)$ .*

Remark: It is also interesting to observe that using the differential equation in  $x$  rather than the one in  $t$  provides a similar set of equations:

$$(B.16) \quad \begin{pmatrix} M^{(k)}(x, t)_{1,1} \\ M^{(k)}(x, t)_{1,2} \\ M^{(k)}(x, t)_{2,1} \end{pmatrix} = \begin{pmatrix} -q_0 & \frac{1}{2} & q_0 \\ x & q_0 & -x \\ -\frac{x+4q_0^2}{2x} & \frac{q_0}{2x} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{x}{(x+2q_0^2)(x-\frac{\theta}{2q_0})} \left( \partial_x M^{(k-1)}(x, t)_{1,1} - \sum_{i=0}^{k-1} [\mathcal{D}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,1} \right) \\ -\frac{x}{(x+2q_0^2)(x-\frac{\theta}{2q_0})} \left( \partial_x M^{(k-1)}(x, t)_{1,2} - \sum_{i=0}^{k-1} [\mathcal{D}^{(k-i)}(x, t), M^{(i)}(x, t)]_{1,2} \right) \\ \frac{\sqrt{2}}{\sqrt{x+2q_0^2}} \sum_{i=1}^{k-1} (M^{(i)}(x, t)_{1,1} M^{(k-i)}(x, t)_{1,1} + M^{(i)}(x, t)_{1,2} M^{(k-i)}(x, t)_{2,1}) \end{pmatrix}.$$

However in this case, it is not possible to exclude pole singularities at the double zero of the spectral curve  $x = \frac{\theta}{2q_0}$ . Thus as in the JM case, we clearly see here the importance of the differential equation with respect to  $t$ .

## APPENDIX C. PROOF OF THE LEADING ORDER OF $W_n$

This section is dedicated to prove that the leading order of the series expansion in  $\hbar$  of  $W_n$  is at least of order  $\hbar^{n-2}$ :

$$(C.1) \quad W_n(x_1, \dots, x_n) = O(\hbar^{n-2}) \quad \text{for } n \geq 1.$$

First we explain why the standard procedure using a so-called insertion operator is incomplete. Then we present a new proof using the structure of the loop equations.

**C.1. Incomplete proof using an insertion operator.** In [2, 3, 20] the various authors presented the construction of an insertion operator  $\delta_\eta$  to prove the leading order of the  $\hbar$  expansion of  $W_n$ . Unfortunately this proof is incomplete and requires an important modification to be correct that is currently being investigated. We present here the main reason for the incompleteness of this method.

The method of the insertion operators naturally applies to the Picard-Vessiot (PV) ring  $\mathbb{B}_1$  attached to the Lax pair, that is to say to the differential ring generated by the entries of  $\Psi(x, t)$  and the scalar function  $\det^{-1} \Psi(x, t)$  over the differential ring  $K_1$  of rational functions of  $x$ . Taking an arbitrary large number of spatial variables, we end up with the projective limit:  $\mathbb{B}_\infty = \bigcup_{i=1}^\infty \mathbb{B}_i$  over the field  $K_\infty = \bigcup_{i=1}^\infty K_i$  of rational functions in any arbitrary large number of variables  $x_i$ . Most quantities defined in this paper belongs to the PV ring  $\mathbb{B}_\infty$  since they can be expressed with the entries of  $\Psi(x, t)$  and  $\det^{-1} \Psi(x, t)$ . For example,  $\Psi^{-1}(x, t)$ ,  $\mathcal{D}(x, t)$ ,  $\mathcal{R}(x, t)$ ,  $K(x_1, x_2)$ ,  $M(x, t)$  and all  $W_n(x_1, \dots, x_n)$  belong to  $\mathbb{B}_\infty$ . The idea of the insertion operator is to create an operator  $\delta_\eta$

acting on the PV ring that satisfy the following properties (i)  $\sim$  (v) (Cf., Definition 2.5, Definition 4.2 and Section 5.7.2 of [2]):

- (i)  $\delta_\eta(K_\infty) = 0$ , and  $\delta_\eta(\mathbb{B}_n) \subset \mathbb{B}_{n+1}$ .
- (ii)  $\delta_\eta$  is a derivation operator:  $\delta_\eta(fg) = (\delta_\eta f)g + f(\delta_\eta g)$ .
- (iii)  $\delta_\eta$  inserts a variable into the correlation functions:

$$(C.2) \quad \delta_\eta W_n(x_1, \dots, x_n) = W_{n+1}(x_1, \dots, x_n, \eta).$$

This property is equivalent to impose that  $\delta_\eta K(x_1, x_2) = -K(x_1, \eta)K(\eta, x_2)$ .

- (iv)  $\delta_\eta M(x, t)$  is of order  $\hbar$ , and is expressed in terms of  $M(x, t)$ ,  $M(\eta, t)$  and their  $t$ -derivatives.
- (v)  $\delta_\eta$  commutes with  $\partial_t$ .

With these properties it is then possible to show that the  $\hbar$  expansion of  $W_n$  must start at least at  $\hbar^{n-2}$ : Firstly, the property (C.2) implies

$$(C.3) \quad W_n(x_1, \dots, x_n) = \delta_{x_n} \cdots \delta_{x_3} W_2(x_1, x_2).$$

On the other hand, the properties (iv) and (v) imply

$$(C.4) \quad \delta_{\eta_1} \cdots \delta_{\eta_n} M(x, t) = O(\hbar^n).$$

Then, since  $W_2(x_1, x_2) = O(\hbar^0)$  is expressed by  $M$  (see (5.9)), we get the desired property (C.1).

In [2] and [20] explicit formulas are proposed for the definition of a suitable insertion operator:

$$(C.5) \quad \delta_\eta \Psi(x, t) = \left( \frac{M(\eta, t)}{x - \eta} + Q(\eta, t) \right) \Psi(x, t),$$

where  $Q(\eta, t)$  is a matrix that depends on the Lax pair and is determined by imposing the above properties. Note that, the insertion operator (C.5) satisfies (C.2) for any choice of  $Q(\eta, t)$ . The property (iv) requires

$$(C.6) \quad \delta_\eta M(x, t) = -\frac{[M(x, t), M(\eta, t)]}{x - \eta} + [Q(\eta, t), M(x, t)] = O(\hbar).$$

Moreover, the condition (v) implies  $[\delta_\eta, \partial_t] \Psi(x, t) = 0$ ; namely,

$$(C.7) \quad \delta_\eta \mathcal{R}(x, t) = \hbar \partial_t Q(\eta, t) + [Q(\eta, t), \mathcal{R}(x, t)] + \frac{[M(\eta, t), \mathcal{R}(\eta, t) - \mathcal{R}(x, t)]}{x - \eta}.$$

The conditions (C.6) and (C.7) almost determine the action of  $\delta_\eta$ . For example in our JM cases we must take:

$$(C.8) \quad Q_{\text{JM}}(\eta, t) = M(\eta, t)_{1,2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$(C.9) \quad \delta_\eta q = 2\hbar \partial_t M(\eta, t)_{1,2}, \quad \delta_\eta p = -2\hbar \partial_t M(\eta, t)_{1,1}.$$

Then, straightforward computations in the JM case shows that:

$$(C.10) \quad \begin{aligned} \delta_\eta M(x, t)_{1,1} &= -\frac{2\hbar}{x - \eta} (M(x, t)_{1,2} \partial_t M(\eta, t)_{1,1} - M(\eta, t)_{1,2} \partial_t M(x, t)_{1,1}), \\ \delta_\eta M(x, t)_{1,2} &= -\frac{2\hbar}{x - \eta} (M(x, t)_{1,2} \partial_t M(\eta, t)_{1,2} - M(\eta, t)_{1,2} \partial_t M(x, t)_{1,2}), \\ \delta_\eta M(x, t)_{2,1} &= \frac{\hbar}{p(x - \eta)} (M(x, t)_{2,1} \partial_t M(\eta, t)_{2,1} - M(\eta, t)_{2,1} \partial_t M(x, t)_{2,1}) \\ &\quad - \frac{2\hbar}{p} M(x, t)_{2,1} \partial_t M(\eta, t)_{1,1}, \\ \delta_\eta M(x, t)_{2,2} &= -\delta_\eta M(x, t)_{1,1}. \end{aligned}$$

Here we have used the following equalities (Cf., (B.2)):

$$\begin{aligned}
\hbar \partial_t M(x, t)_{1,1} &= \mathcal{R}_{1,2} M_{2,1} - M_{1,2} \mathcal{R}_{2,1} = \frac{1}{2} M(x, t)_{2,1} + p M(x, t)_{1,2}, \\
\hbar \partial_t M(x, t)_{1,2} &= 2 \mathcal{R}_{1,1} M_{1,2} - 2 M_{1,1} \mathcal{R}_{1,2} + \mathcal{R}_{1,2} = (x + q) M(x, t)_{1,2} - M(x, t)_{1,1} + \frac{1}{2}, \\
\hbar \partial_t M(x, t)_{2,1} &= 2 \mathcal{R}_{2,1} M_{1,1} - 2 M_{2,1} \mathcal{R}_{1,1} - \mathcal{R}_{2,1} = -2p M(x, t)_{1,1} - (x + q) M(x, t)_{2,1} + p, \\
\hbar \partial_t M(x, t)_{2,2} &= -\hbar \partial_t M(x, t)_{1,1}.
\end{aligned}$$

From the formula (C.10) it is tempting to conclude that the insertion operator  $\delta_\eta$  satisfies the property (iv).

However (C.10) is not sufficient to prove (C.4) due to the following reason. Since the r.h.s. of (C.10) involves time derivatives of  $M(x, t)$ , iterative application of the insertion operators will create terms of the form  $\delta_\eta \partial_t M(x, t)$ ,  $\delta_\eta \partial_t^2 M(x, t)$  and so on. The problem is that the condition (C.7) is not enough to prove  $[\delta_\eta, \partial_t] = 0$  as operators acting on PV ring. For example, the condition (C.7) is not enough to prove  $[\delta_\eta, \partial_t^2] \Psi(x, t) = 0$ . To have this identity, we also need to require

$$(C.11) \quad [\delta_\eta, \partial_t] \mathcal{R}(x, t) = 0$$

and this cannot hold for JM Lax pair (Cf., (C.9)). Consequently this method does not prove (C.4) and (C.1).

Unfortunately this problem is not specific to the JM Lax pair and is really intrinsic to the current method of the insertion operator. For example it also arises in the HTW Lax pair as well. We could not find any simple way to fix the problem and it is likely that substantial modifications of the insertion operator are required. However since the insertion operator exists in the context of random matrix models we believe that it should exist in the context of determinantal formulas too. Anyway, we present in the next section another method that does not require the use of an insertion operator.

**C.2. Proof using loop equations.** Determinantal formulas (5.1) have been introduced so that they satisfy a set of equations known as the loop equations. These loop equations (also known as Schwinger-Dyson equations) originate in random matrix theory where they are crucial. We recall here the main result of [3]:

**Proposition C.1** (Theorem 2.9 of [3]). *Let us define the following functions (we denote by  $L_n$  the set of variables  $\{x_1, \dots, x_n\}$ ):*

$$\begin{aligned}
(C.12) \quad P_1(x) &= \frac{1}{\hbar^2} \det \mathcal{D}(x, t), \\
P_2(x; x_2) &= \frac{1}{\hbar} \text{Tr} \left( \frac{\mathcal{D}(x, t) - \mathcal{D}(x_2, t) - (x - x_2) \mathcal{D}'(x_2, t)}{(x - x_2)^2} M(x_2) \right), \\
P_{n+1}(x; L_n) &= (-1)^n \left[ Q_{n+1}(x; L_n) - \sum_{j=1}^n \frac{1}{x - x_j} \text{Res}_{x' \rightarrow x_j} Q_{n+1}(x', L_n) \right], \\
Q_{n+1}(x; L_n) &= \frac{1}{\hbar} \sum_{\sigma \in S_n} \frac{\text{Tr} (\mathcal{D}(x) M(x_{\sigma(1)}) \dots M(x_{\sigma(n)}))}{(x - x_{\sigma(1)})(x_{\sigma(1)} - x_{\sigma(2)}) \dots (x_{\sigma(n-1)} - x_{\sigma(n)})(x_{\sigma(n)} - x)}.
\end{aligned}$$

Then, the correlation functions satisfy

$$(C.13) \quad P_1(x) = W_2(x, x) + W_1(x)^2,$$



and

$$(C.14) \quad 0 = P_{n+1}(x; L_n) + W_{n+2}(x, x, L_n) + 2W_1(x)W_{n+1}(x, L_n) \\ + \sum_{J \subset L_n, J \notin \{\emptyset, L_n\}} W_{1+|J|}(x, J)W_{1+n-|J|}(x, L_n \setminus J) + \sum_{j=1}^n \frac{d}{dx_j} \frac{W_n(x, L_n \setminus x_j) - W_n(L_n)}{x - x_j} \quad \text{for } n \geq 1.$$

Moreover  $P_{n+1}(x; L_n)$  is a rational function of  $x$  whose poles are at the poles of  $\mathcal{D}(x, t)$ .

The equations (C.13) and (C.14) are called the loop equations. As we will see this proposition and a subtle induction are sufficient to prove that  $W_n$  is at least of order  $\hbar^{n-2}$ . Let us now make the following crucial observation:

**Theorem C.1.** *In the JM Lax pair case,  $P_{n+1}(x; L_n)$  does not depend on  $x$  for  $n \geq 1$ . In the HTW Lax pair case, the functions  $P_{n+1}(x; L_n)$  are of the form  $P_{n+1}(x; L_n) = \frac{\tilde{P}_{n+1}(L_n)}{x}$ .*

Proof: In the JM case, since the entries of  $\mathcal{D}(x, t)$  is polynomial of  $x$ ,  $P_{n+1}(x; L_n)$  may only have singularity at  $x = \infty$ . However looking at large  $x$  the definition shows that  $P_{n+1}(x; L_n)$  can only be a polynomial of degree 0 and hence does not depend on  $x$ . In addition we get an explicit formula:

$$(C.15) \quad \forall n \geq 1 : P_{n+1}(x; L_n) = P_{n+1}(L_n) = \frac{(-1)^{n+1}}{\hbar} \sum_{\sigma \in S_n} \frac{\text{Tr}(\sigma_3 M(x_{\sigma(1)}) \dots M(x_{\sigma(n)}))}{(x_{\sigma(1)} - x_{\sigma(2)}) \dots (x_{\sigma(n-1)} - x_{\sigma(n)})}.$$

For example, using directly the definition we have  $P_2(x; x_2) = \frac{1}{\hbar} \text{Tr}(\sigma_3 M(x_2))$  which is indeed independent of  $x$ . In the HTW case, the form of  $\mathcal{D}(x, t)$  implies that  $P_{n+1}(x; L_n)$  may only have simple poles at  $x = 0$  and a simple zero at infinity (degree of numerator-denominator shows that it behaves as  $O(\frac{1}{x})$  at infinity). Hence we conclude that it is proportional to  $\frac{1}{x}$  (one could even get a complete expression by taking the residue at  $x = 0$ ).

Additionally we observe that by construction  $W_1^{(-1)}(x)$  corresponds to the spectral curve of the problem. Indeed, let us look the first loop equation (C.13). Since  $W_2(x_1, x_2)$  only starts at  $\hbar^0$  (obvious from its definition) while  $P_1(x) = \frac{1}{\hbar^2} \det \mathcal{D}$  looking at order  $\hbar^{-2}$  in the last equation provides the result. We now have all the ingredients to prove the following theorem:

**Theorem C.2.** *The correlation functions  $W_n(x_1, \dots, x_n)$  admit a series expansion in  $\hbar$  starting at least at order  $\hbar^{n-2}$ .*

The proof is done by induction. We will denote  $L_i = \{x_1, \dots, x_i\}$ . Let us define the following statement:

$$(C.16) \quad \mathcal{P}_k : W_j(x_1, \dots, x_j) \text{ is at least of order } \hbar^{k-2} \quad \text{for } j \geq k.$$

The statement is obviously true for  $k = 1$  and  $k = 2$ . Let us assume that the statement  $\mathcal{P}_i$  is true for all  $i \leq n$ . Now we look at the loop equation (C.14). By induction assumption, we have that the last two terms are at least of order  $\hbar^{n-2}$ . Indeed in the sum we have terms of order  $\hbar^{1+|J|-2+1+n-|J|-2} = \hbar^{n-2}$ . Moreover we also have from the same assumption that  $W_{n+2}(x, x, L_n)$  is also of order at least  $\hbar^{n-2}$  (since  $n+2 \geq n$ ). Eventually  $W_{n+1}(x, L_n)$  is at least of order  $\hbar^{n-2}$  so we get when looking at order  $\hbar^{n-3}$  in (C.14):

$$(C.17) \quad 0 = P_{n+1}^{(n-3)}(x; L_n) + 2W_1^{(-1)}(x)W_{n+1}^{(n-2)}(x, L_n).$$

If we assume that  $W_{n+1}^{(n-2)}(x, L_n) \neq 0$  then we have:

$$(C.18) \quad W_{n+1}^{(n-2)}(x, L_n) = \frac{P_{n+1}^{(n-3)}(x; L_n)}{2W_1^{(-1)}(x)}.$$

In our cases we get:

$$(C.19) \quad \begin{aligned} \text{JM} : W_{n+1}^{(n-2)}(x, L_n) &= \frac{P_{n+1}^{(n-3)}(L_n)}{2(x - q_0)\sqrt{(x + q_0)^2 + \frac{\theta}{q_0}}}, \\ \text{HTW} : W_{n+1}^{(n-2)}(x, L_n) &= \frac{\tilde{P}_{n+1}^{(n-3)}(L_n)}{2x} \frac{2x}{\left(x - \frac{\theta}{2q_0}\right)\sqrt{x + 2q_0^2}} = \frac{\tilde{P}_{n+1}^{(n-3)}(L_n)}{\left(x - \frac{\theta}{2q_0}\right)\sqrt{x + 2q_0^2}}. \end{aligned}$$

In both cases we obtain that  $W_{n+1}^{(n-2)}(x, L_n)$  must have a simple pole at the even zero of the spectral curve which contradicts the pole structure B.1 or B.2. Consequently we must have  $W_{n+1}^{(n-2)}(x, L_{i_0}) = 0$ . This proves that  $W_{n+1}(x, L_n)$  is at least of order  $\hbar^{n-1}$ . We now need to prove the same statement for higher correlation functions. Let us prove it by a second induction by defining:

$$(C.20) \quad \tilde{\mathcal{P}}_i : W_i(x_1, \dots, x_i) \text{ is of order at least } \hbar^{n-1}.$$

We want to prove  $\tilde{\mathcal{P}}_i$  for all  $i \geq n+1$  by induction. We just proved it for  $i = n+1$  so initialization is done. Let us assume that  $\tilde{\mathcal{P}}_j$  is true for all  $j$  satisfying  $n+1 \leq j \leq i_0$ . We look at the loop equation:

$$(C.21) \quad \begin{aligned} 0 &= P_{i_0+1}(x; L_{i_0}) + W_{i_0+2}(x, x, L_{i_0}) + 2W_1(x)W_{i_0+1}(x, L_{i_0}) \\ &+ \sum_{J \subset L_{i_0}, J \notin \{\emptyset, L_{i_0}\}} W_{1+|J|}(x, J)W_{1+i_0-|J|}(x, L_{i_0} \setminus J) \\ &+ \sum_{j=1}^{i_0} \frac{d}{dx_j} \frac{W_{i_0}(x, L_{i_0} \setminus x_j) - W_{i_0}(L_{i_0})}{x - x_j}. \end{aligned}$$

By assumption on  $\tilde{\mathcal{P}}_{i_0}$ , the last sum with the derivatives contains terms of order at least  $\hbar^{n-1}$ . In the sum involving the subsets of  $L_{i_0}$  it is straightforward to see that the terms are all of order at least  $\hbar^{n-1}$ . Indeed, as soon as one of the index is greater than  $n+1$ , the assumption of  $\tilde{\mathcal{P}}_i$  for  $n+1 \leq i \leq i_0$  tells us that this term is already at order at least  $\hbar^{n-1}$ . Since the second factor of the product is at least of order  $\hbar^0$  then it does not decrease the order. Now if both factors have indexes strictly lower than  $n+1$ , then the assumption of  $\mathcal{P}_j$  for all  $j \leq n$  tell us that the order of the product is at least of  $\hbar^{|J|+1-2+1+i_0-|J|-2} = \hbar^{i_0-2}$  which is greater than  $n-1$  since  $i_0 \geq n+1$ . Additionally by induction on  $\mathcal{P}_n$  we know that  $W_{i_0+1}(x, L_{i_0})$  is at least of order  $\hbar^{n-2}$  as well as  $W_{i_0+2}(x, x, L_{i_0})$ . Consequently looking at order  $\hbar^{n-3}$  in (C.21) gives:

$$(C.22) \quad 0 = P_{i_0+1}^{(n-3)}(x; L_{i_0}) + 2W_1^{(-1)}(x)W_{i_0+1}^{(n-2)}(x, L_{i_0}).$$

We can apply a similar reasoning as for (C.17). If we assume that  $W_{i_0+1}^{(n-2)}(x, L_{i_0}) \neq 0$  then we have:

$$(C.23) \quad W_{i_0+1}^{(n-2)}(x, L_{i_0}) = \frac{P_{i_0+1}^{(n-3)}(x; L_{i_0})}{2W_1^{(-1)}(x)}.$$

In our two cases we get:

$$(C.24) \quad \begin{aligned} \text{JM} : W_{i_0+1}^{(n-2)}(x, L_{i_0}) &= \frac{P_{i_0+1}^{(n-3)}(L_{i_0})}{2(x - q_0)\sqrt{(x + q_0)^2 + \frac{\theta}{q_0}}}, \\ \text{HTW} : W_{i_0+1}^{(n-2)}(x, L_{i_0}) &= \frac{\tilde{P}_{i_0+1}^{(n-3)}(L_{i_0})}{2x} \frac{2x}{\left(x - \frac{\theta}{2q_0}\right)\sqrt{x + 2q_0^2}} = \frac{\tilde{P}_{i_0+1}^{(n-3)}(L_{i_0})}{\left(x - \frac{\theta}{2q_0}\right)\sqrt{x + 2q_0^2}}. \end{aligned}$$

In both cases we obtain that  $W_{i_0+1}^{(n-2)}(x, L_{i_0})$  must have a simple pole at the even zero of the spectral curve which contradicts the pole structure B.1 or B.2. Consequently we must have  $W_{i_0+1}^{(n-2)}(x, L_{i_0}) = 0$ . In particular it means that  $W_{i_0+1}(x, L_{i_0})$  (which by assumption of  $\mathcal{P}_n$  was already known to be of order  $\hbar^{n-2}$ ) is at least of order  $\hbar^{n-1}$  thus making the induction on  $\tilde{\mathcal{P}}_{i_0}$ . Hence by induction we have proved that  $\forall i \geq n+1$ ,  $\tilde{\mathcal{P}}_i$  holds which exactly proves that  $\mathcal{P}_{n+1}$  is true. Eventually by induction we have just proved that  $\mathcal{P}_n$  holds for  $n \geq 1$ , which implies the desired equality (C.1).

Remarks: Several important observations can be made about this proof:

- The proof heavily relies on the pole structure of the correlation functions  $W_n^{(g)}$ . In particular it is central to know that the correlation functions are regular at the even zeros of the spectral curve since it provides the contradiction in (C.17) and (C.22).
- The possible poles of  $\mathcal{D}(x, t)$  are irrelevant in the proof. Indeed, they specify the form of  $P_{n+1}(x; L_n)$  but do not play an important role in the contradiction of (C.17) and (C.22).
- The presence of at least one even zero in the spectral curve is necessary in our proof because  $\mathcal{D}(x, t)$  is a polynomial of degree 2 in the JM case or has a simple pole at  $x = 0$  in the HTW case. In the case when  $\mathcal{D}(x, t)$  is a polynomial of degree 1, then  $P_{n+1}(x; L_n)$  would be identically zero and thus our proof would also work in a simpler way. Hence the central element is the balance between the order of the singularity of  $\mathcal{D}(x, t)$  and the fact that the spectral curve is of genus 0.
- This proof can be applied to more general cases: as soon as the spectral curve has a double zero and the pole structure is proved then the method can be applied. In the case of Lax pairs, the pole structure is usually easy to obtain from the  $t$ -differential equation like we did in appendix B and the spectral curve is even simpler to obtain.

#### APPENDIX D. TAU-FUNCTION AND SYMPLECTIC INVARIANTS

Here we show that the (generating function of) symplectic invariants give a tau-function of Painlevé 2, following the idea of [2, 7].

Let  $W_1(x)$  be given in (5.7). Direct computation shows that

$$(D.1) \quad W_1(x) = \begin{cases} \frac{x^2}{\hbar} + \frac{t}{2\hbar} - \frac{\theta}{\hbar x} + \frac{\sigma(t)}{\hbar x^2} + O(x^{-2}) & \text{for JM case,} \\ \frac{x^{\frac{1}{2}}}{\sqrt{2}\hbar} - \frac{tx^{-\frac{1}{2}}}{2\sqrt{2}\hbar} - \frac{\sigma(t) + \frac{t^2}{8}}{\sqrt{2}\hbar} x^{-\frac{3}{2}} + O(x^{-2}) & \text{for HTW case.} \end{cases}$$

when  $x \rightarrow \infty$ . Then, we have

$$(D.2) \quad 2 \operatorname{Res}_{x=\infty} \left( \frac{1}{\hbar} \frac{\partial s_\infty}{\partial t}(x) W_1(x) dx \right) = \begin{cases} \frac{d}{dt} \ln \tau_{\text{JM}}(t, \hbar) & \text{for JM case,} \\ \frac{d}{dt} \ln \tau_{\text{HTW}}(t, \hbar) & \text{for HTW case.} \end{cases}$$

where  $s_\infty(x)$  is the divergent part of  $s(x)$  given in (5.3) when  $x \rightarrow \infty$ :

$$(D.3) \quad s_\infty(x) = \begin{cases} \frac{x^3}{3} + \frac{tx}{2} & \text{for JM case,} \\ \frac{\sqrt{2}x^{\frac{3}{2}}}{3} - \frac{tx^{\frac{1}{2}}}{\sqrt{2}} & \text{for HTW case.} \end{cases}$$

The previous quantities give an alternative definition of the tau-function in terms of the solution of the isomonodromy system ([15, §5]; see also [2, §4.2] and [7, §1.5]). Then, the following theorem relates the tau-function (D.2) with the symplectic invariants.

**Theorem D.1** (Theorem 5.1 of [12]). *For  $g \geq 1$ , both  $F^{(g)} = F_{\text{JM}}^{(g)}$  and  $F_{\text{HTW}}^{(g)}$  satisfy*

$$(D.4) \quad \frac{dF^{(g)}}{dt} = 2 \operatorname{Res}_{x=\infty} \left( \frac{\partial s_\infty}{\partial t}(x) W_1^{(g)}(x) dx \right).$$

Proof: The results of Appendix B, A and C and Proposition 5.1 imply

$$(D.5) \quad W_1^{(g)}(x(z)) dx(z) = \omega_1^{(g)}(z),$$

where  $x(z)$  appears in the parametrization (3.22) or (4.9) of the spectral curve, and  $\omega_1^{(g)}(z)$  is the Eynard-Orantin differential of type  $(g, 1)$ . On the other hand, the function  $\Lambda(z) = \frac{\partial s_\infty}{\partial t}(x(z))$  satisfies

$$\frac{\partial x}{\partial t}(z) dy(z) - \frac{\partial y}{\partial t}(z) dx(z) = \begin{cases} \operatorname{Res}_{w=\infty} \Lambda(w) \omega_2^{(0)}(w, z) - \operatorname{Res}_{w=0} \Lambda(w) \omega_2^{(0)}(w, z) & \text{for JM case,} \\ \operatorname{Res}_{w=\infty} \Lambda(w) \omega_2^{(0)}(w, z) & \text{for HTW case.} \end{cases}$$

This equation is the required condition for  $\Lambda(z)$  to apply Theorem 5.1 of [12], and thus we have (D.4). This completes the proof of Theorem 3.1 and Theorem 4.1.

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